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THÈSE

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BOUZIDI Louanas

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Thème

Etude de quelques problèmes aux limites dans des domaines non réguliers

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Devant le Jury composé de :

Nom et Prénom	Grade		
Mr BOUKOUCHA Rachid	Professeur	Univ. de Bejaia	Président
Mr KHELOUFI Arezki	Professeur	Univ. de Bejaia	Rapporteur
Mr HALIM Yacine	MCA	Centre Univ. de Mila	Examinateur
Mr BOUDREF Mohamed Ahmed	MCA	Univ. Bouira	Examinateur

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ABSTRACT

This thesis is designed to give some results of existence and uniqueness of solutions for some parabolic equations posed in unbounded in time non-cylindrical domains. We give sufficient conditions on the functions of the parametrization of the non regular domains and on the coefficients of the equations under which our problems admit unique solutions. We study the global regularity problem in a suitable parabolic Sobolev space. The method used to prove our main results is based on the technique of the decomposition of domains.

Key words. parabolic equations, heat equation, non-rectangular domains, conical domains, unbounded domains, Dirichlet-Robin conditions, anisotropic Sobolev spaces.

RÉSUMÉ

Cette thèse a pour but de donner des résultats d'existence et d'unicité de solutions pour certaines équations paraboliques posées dans des domaines non cylindriques et non bornés en temps. Nous donnons des conditions suffisantes sur les fonctions de paramétrisation des domaines non réguliers et sur les coefficients des équations sous lesquelles nos problèmes admettent une solution unique. Nous étudions le problème de régularité globale dans un espace de Sobolev parabolique approprié. La méthode utilisée pour démontrer nos principaux résultats est basée sur la technique de la décomposition de domaines.

Mots clés. Equations paraboliques, équation de la chaleur, domaines non rectangulaires, domaines coniques, domaines non bornés, conditions de Dirichlet-Robin, espaces de Sobolev anisotropes.

CONTENTS

INTRODUCTION

In this thesis, we shall be concerned by the existence, the uniqueness and maximal regularity of solutions of the following equation :

$$\partial_t u - c(t)\Delta u = f, \tag{1}$$

where Δ is the Laplacian operator and $c(t)$ is a time dependent coefficient. The second member f belongs to the Lebesgue space of square integrable functions L^2. The equation (1) is posed in unbounded time-varying domains of $\mathbb{R}^n, n = 2$ or $n = 3$ and it is associated with boundary conditions of the Dirichlet-Robin type or of the Cauchy-Dirichlet type.

The main pecularities of the problems studied in this thesis are the unboundedness of the domains and the fact that they are non regular. Besides being interesting in themselves, such kind of problems are of interest in several fields, see for example [6], [16] and the references therein.

This thesis consists of three chapters. Let us briefly indicate the contents of each chapter.

Chapter 1 is a preliminary chapter in which we recalled essential notions and results that will be used throughout this work. First, we recall some definitions concerning some functional spaces, notably the anisotropic Sobolev spaces. We then prove some technical lemmas. Finally, we present some results on some model parabolic problems that we need to develop further arguments.

In Chapter 2, we will prove well-posedness and regularity results for a one-dimensional parabolic equation, subject to Dirichlet-Robin type boundary conditions and posed in an unbounded in time non-rectangular domain. More precisely, we are concerned by the following problems :

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f_1 \in L^2(\Omega), \\ u|_{\Gamma_1} = 0, \\ \partial_x u + \beta_2 u|_{\Gamma_2} = 0, \end{cases} \tag{2}$$

and

$$\begin{cases} \partial_t v - c(t)\partial_x^2 v = f_2 \in L^2(\Omega), \\ v|_{\Gamma_2} = 0, \\ \partial_x v + \beta_1 v|_{\Gamma_1} = 0, \end{cases} \tag{3}$$

where

$$\Omega := \left\{(t,x) \in \mathbb{R}^2 : t > 0,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

with $\varphi_i \in C([0,+\infty[) \cap C^1(0,+\infty), i = 1,\ 2,$

$$\varphi(t) := \varphi_2(t) - \varphi_1(t) > 0 \quad \forall t > 0, \text{ and } \varphi(0) = 0.$$

The lateral boundaries of Ω are defined by

$$\Gamma_i = \left\{(t, \varphi_i(t)) \in \mathbb{R}^2 : t > 0\right\},\ i = 1,\ 2.$$

The coefficient c is a continuous real-valued function defined on $[0,+\infty[$, differentiable on $]0,+\infty[$ and such that

$$0 < \alpha \leq c(t) \leq \beta$$

for every $t \in [0,+\infty[$, where α and β are positive constants. Here, the coefficient $\beta_i,\ i = 1,\ 2$ in boundary conditions are real numbers.

Problems (2) and (3) modelize, for instance, the lateral diffusion of two pollutants in a flow of a river with variable width. Note that the Robin type conditions

$$\partial_x v + \beta_1 v|_{\Gamma_1} = \partial_x u + \beta_2 u|_{\Gamma_2} = 0,$$

mean for instance, that the flux of diffusion of the pollutants are proportional to their propagations along the wide of the river. The most interesting points of the parabolic problems studied here is the unboundedness of D with respect to the time variable t and the fact that D shrinks at $t = 0$ ($\varphi(0) = 0$) which prevent one using the methods in [20] and [21]. It is well known that there are two main approaches for the study of boundary value problems in such non-regular domains. The analysis can be done in weighted spaces with the weight controlling the behavior of the solutions near the singularity of the boundary of the domain (see, for instance, [17], [18] and [19]). Our approach is different. Indeed, the space

$$\mathcal{H}^{1,2}(\Omega) := \left\{ w \in L^2(\Omega) : \partial_t w, \partial_x w, \partial_x^2 w \in L^2(\Omega) \right\}$$

used here has low smoothness but one must add assumptions on the type of the domain Ω, as well as conditions on the coefficients c and $\beta_i,\ i = 1,\ 2$, near the singular point 0 and in the neighborhood of $+\infty$.

In Chapter 3, we will prove well-posedness and regularity results for a bi-dimensional parabolic equation, subject to Cauchy-Dirichlet boundary conditions and posed in an unbounded in time conical domain of $\mathbb{R}^3$. More precisely, we are concerned by the following problem :

$$\begin{cases} \partial_t w - c(t)(\partial_x^2 w + \partial_y^2 w) = f \in L^2(D), \\ w|_{\partial D} = 0, \end{cases} \tag{4}$$

where the conical domain D is defined by

$$D = \left\{ (t, x, y) \in \mathbb{R}^3 : t > 0;\ 0 \leq \sqrt{x^2 + y^2} < \varphi(t) \right\}$$

where $\varphi \in C([0, +\infty[) \cap C^1(0, +\infty)$,

$$\varphi(t) > 0 \quad \forall t > 0, \text{ and } \varphi(0) = 0.$$

The coefficient c is a continuous real-valued function defined on $[0, +\infty[$, differentiable on $]0, +\infty[$ and such that

$$0 < \gamma \leq c(t) \leq \delta$$

for every $t \in [0, +\infty[$, where γ and δ are positive constants. by using the domain decomposition method, we prove that there exists a unique solution w of Problem (4) belonging to the anisotropic Sobolev space

$$\mathcal{H}^{1,2}(D) = \left\{ w \in H^1(D) : \partial_x^2 w \in L^2(D), \partial_y^2 w \in L^2(D), \partial_{xy}^2 w \in L^2(D) \right\}$$

where $H^1(D)$ stands for the Sobolev space defined by

$$H^1(D) = \left\{ w \in L^2(D) : \partial_t w \in L^2(D), \partial_x w \in L^2(D), \partial_y w \in L^2(D) \right\}$$

Note that results concerning the bounded in time case are obtained for Problem (4) in [13].

We end this thesis by a conclusion and prospects.

CHAPTER 1

PRELIMINARIES

The objective of this chapter is to recall the essential notions and results used throughout this work. First, we recall some definitions concerning some functional spaces, notably the anisotropic Sobolev spaces. We then prove some technical lemmas. Finally, we present some results on some model parabolic problems. For more details, see [5], [8] and [21].

1.1 Some functional spaces

1.1.1 L^p spaces

Definition 1.1.1. *Let Ω be an open subset of $\mathbb{R}^N$ and $p \in [1, +\infty]$. The space L^p is the vectorial space of classes of functions u from Ω into $\mathbb{R}$, Lebesgue measurables, such that*

1. *if $1 \leq p < +\infty$, $\displaystyle\int_\Omega |u(x)|^p \; dx < +\infty$,*
2. *if $p = +\infty$, $\underset{x\in\Omega}{ess\sup}|u(x)| < +\infty$, where*

$$\underset{x\in\Omega}{ess\sup}|u(x)| = \inf\{M| \; |u(x)| \leq M \;\; a. \;\; e. \;\; x \in \Omega\}.$$

Proposition 1.1.1. *1. The mapping $\|\cdot\|$ defined from $\mathrm{L}^p(\Omega)$ into $\mathbb{R}_+$ by*

$$u \longmapsto \begin{cases} \|u\|_p = (\displaystyle\int_\Omega |u(x)|^p \; dx)^{\frac{1}{p}}, & 1 \leq p < +\infty, \\ \|u\|_\infty = \underset{x\in\Omega}{supess}|u(x)|, & p = \infty, \end{cases}$$

define a norm on $\mathrm{L}^p(\Omega)$*, which makes it a Banach space (and a Hilbert space if* $p = 2$*).*

2. *For every real* $p \in [1, +\infty[$*, the dual* $\mathrm{L}^{p'}(\Omega)$ *of* $\mathrm{L}^p(\Omega)$ *is isomorph to* $\mathrm{L}^p(\Omega)$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$*. The dual mapping is defined by*

$$\mathrm{L}^p(\Omega) \times \mathrm{L}^{p'}(\Omega) \longmapsto \mathbb{R},\ (u, v) \longmapsto \int_\Omega u(x)v(x)\ dx,$$

for each real $p \in [1, +\infty[$*. The bi-dual of* $\mathrm{L}^p(\Omega)$ *can be identified to* $\mathrm{L}^p(\Omega)$*. We said that* $\mathrm{L}^p(\Omega)$ *is a reflexive space.*

Theorem 1.1.1. ***(Hölder inequality)***
Assume that $f \in \mathrm{L}^p$ *and* $g \in \mathrm{L}^{p'}$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$*. Then* $fg \in \mathrm{L}^1$ *and*

$$\int_\Omega |fg|\ dx \leq \|f\|_{\mathrm{L}^p}\|g\|_{\mathrm{L}^{p'}}.$$

We can find the proof of the previous inequality, for example, in [5] (Theorem 4.6 page 50).

Remark 1.1.1. *If* $p = p' = 2$*, then we obtain the Cauchy-Schwarz inequality*

$$\int_\Omega |fg|\ dx \leq \|f\|_{\mathrm{L}^2}\|g\|_{\mathrm{L}^2}.$$

1.1.2 Sobolev spaces

Assume that Ω is an open domain in $\mathbb{R}^N$. For $m \in \mathbb{N}$ and $p \in [1, +\infty[$, the Sobolev space $W^{m,p}(\Omega)$ is defined by :

$$W^{m,p}(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} | D^\alpha u \in L^p(\Omega), \forall \alpha \in \mathbb{N}^N, 0 \leq |\alpha| \leq m\}$$

where for any $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ we note $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_N$ and $D^\alpha u = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}...\partial^{\alpha_N}_{x_N}u$. The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^p_{\mathrm{L}^p(\Omega)} \tag{1.1}$$

or with the equivalent norm

$$\|u\|_{W^{m,p}(\Omega)} = (\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^p_{\mathrm{L}^p(\Omega)})^{\frac{1}{p}}.$$

Proposition 1.1.2. *$W^{m,p}(\Omega)$ is a separable Banach space. It is reflexive for $1 < p < \infty$.*

Remark 1.1.2. 1. *If $p = 2$, we usually write $W^{m,2}(\Omega) = \mathcal{H}^m(\Omega)$. When equipped with the inner product*

$$(u|v)_{\mathcal{H}^m(\Omega)} = \sum_{0\le|\alpha|\le m} (D^\alpha u|D^\alpha v)_{\mathrm{L}^2(\Omega)},$$

$\mathcal{H}^m(\Omega)$ is a Hilbert space.

2. *If Ω is an open bounded set with smooth boundary Γ, the the norm* (1.1) *is equivalente to the following norm*

$$\|u\|_{\mathrm{L}^p(\Omega)} + \sum_{|\alpha|\le m} \|D^\alpha u\|_{\mathrm{L}^p(\Omega)}.$$

Definition 1.1.2. *$D(\Omega)$ is the set of functions of class $C^\infty(\Omega)$ with compact support in Ω, i. e.*

$$\mathrm{D}(\Omega) = \{u : \Omega \to \mathbb{R};\ u \in C^\infty(\Omega) \text{ et } supp(u) \subset K \subset \Omega,\ K \text{ compact}\}.$$

Definition 1.1.3. *We note by $W_0^{m,p}(\Omega) = \overline{\mathrm{D}(\Omega)}^{W^{m,p}(\Omega)}$ the closure of de $\mathrm{D}(\Omega)$ in $W^{m,p}(\Omega)$.*

Theorem 1.1.2. *$D(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$.*

In the sequel, $W^{1,p}(\Omega)$ will be equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = (\|u\|^p_{\mathrm{L}^P(\Omega)} + \sum_{j=1}^{N} \|\frac{\partial u}{\partial x_j}\|^p_{\mathrm{L}^P(\Omega)})^{\frac{1}{p}}$$

or with the equivalent norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{\mathrm{L}^p(\Omega)} + \sum_{j=1}^{N} \|\frac{\partial u}{\partial x_j}\|_{\mathrm{L}^p(\Omega)}.$$

In the case of a bounded set Ω, $W^{1,p}(\Omega)$ can be equipped with the following norm:

$$\|u\|_{W_0^{1,p}(\Omega)} = (\int_\Omega |\nabla u|^p\ dx)^{\frac{1}{p}}$$

which is called the gradient norm. The equivalence of these norms can be obtained by using the following result :

Theorem 1.1.3. ***(Poincaré Inequality)***
Assume that Ω is bounded. Then, there exists a constant C (which depends on Ω and on p) such that

$$\|u\|_{\mathrm{L}^p(\Omega)} \leq C\|\nabla u\|_{\mathrm{L}^p(\Omega)}, \; \forall u \in W_0^{1,p}(\Omega), \; 1 \leq p < \infty.$$

The mapping $u \longmapsto \|\nabla u\|_{\mathrm{L}^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$ which is equivalente to the norm induced by $\|\cdot\|_{W^{1,p}(\Omega)}$.

Theorem 1.1.4. ***(Green Formula)***
Assume that Ω is a open bounded set of $\mathbb{R}^N$ of class C^1 such that its boundary is bounded. Then, for each $u \in \mathcal{H}^2(\Omega)$ and for each $v \in \mathcal{H}^1(\Omega)$, we have

$$\int_\Omega \Delta u \cdot v \; dx = -\int_\Omega \nabla u \cdot \nabla v \; dx + \int_\Gamma \frac{\partial u}{\partial n} \cdot v \; d\sigma$$

where $\frac{\partial u}{\partial n} = \nabla u \cdot n = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \eta_i$ is the normal derivative and $n = {}^t(n_1, \ldots, n_N)$ is the unit normal vector.

The Green formula can be given by the following theorem :

Theorem 1.1.5. ***(Green Formula)***
Assume that Ω is an open bounded domain of $\mathbb{R}^N$ with Lipschitz boundary $\Gamma = \partial\Omega$ and $1 < p < \infty$. Then, for every $u \in W^{1,p}(\Omega)$ and for every $v \in W^{-1,p'}(\Omega)$, we have

$$\int_\Omega \frac{\partial u}{\partial x_i} v \; dx \;\; = \;\; -\int_\Omega u \frac{\partial v}{\partial x_i} \; dx + \int_\Gamma uv v_i \; d\sigma, \; i = 1 \ldots N, \tag{1.2}$$

where $\frac{\partial u}{\partial x_i}$ and $\frac{\partial v}{\partial x_i}$ are taken in the distributional sens, v_i is the i-th composante of unit vector of the normale to $\partial\Omega$. Here, $W^{-1,p'}(\Omega)$ denotes the dual space of $W^{1,p}(\Omega)$, $1 < p < \infty$.

1.1.3 Anisotropic Sobolev spaces

It is well known that Lebesgue and Sobolev spaces are the essential tools of functional analysis for the study of partial differential equations. The study of the regularity of solutions can be carried out by their belonging to some functional spaces. In this section, we introduce the so-called anisotropic Sobolev spaces $\mathcal{H}^{1,2}$ built on the Lebesgue space of square integrable functions L^2. These spaces are the natural ones adopted in the study of second-order parabolic equations. The main features of the spaces $\mathcal{H}^{1,2}$ is that smoothness with respect to spatial variables are twice as high with respect to time. For more details and proofs we refer to Lions and Magenes [21].

Let us introduce some notations. Let Ω be a bounded open subset of $\mathbb{R}^n$. We denote $Q = (0,T) \times \Omega$ with T maybe finite or infinite. $L^2(Q)$ denotes the space of (class of) functions u which are square integrable on Q with the Lebesgue measure $dtdx$, such that

$$\|u\|_{L^2(Q)} = \left(\int_Q |u|^2 \, dtdx \right)^{1/2} < \infty.$$

Now we define the anisotropic Sobolev space $\mathcal{H}^{1,2}(Q)$

Definition 1.1.4. $\mathcal{H}^{1,2}(Q) = \{u \in L^2(Q),\ \partial_t u \in L^2(Q),\ \partial^\alpha u \in L^2(Q),\ \textit{for each } |\alpha| \leq 2\}$.

$\partial^\alpha u = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n \leq 2$. Of course, it is a Hilbert space with inner product

$$(u,v)_{\mathcal{H}^{1,2}(Q)} = (\partial_t u, \partial_t v)_{L^2(Q)} + \sum_{|\alpha| \leq 2} (\partial^\alpha u, \partial^\alpha v)_{L^2(Q)}$$

$\mathcal{H}^{1,2}(Q)$ is equipped with the norm

$$\|u\|_{\mathcal{H}^{1,2}(Q)} = \left(\|\partial_t u\|^2_{L^2(Q)} + \sum_{|\alpha| \leq 2} \|\partial^\alpha u\|^2_{L^2(Q)} \right)^{1/2}.$$

Remark 1.1.3. *The spaces $H^{1,2}(Q)$, are said to be anisotropic in the sense that orders of differentiability in the time and spatial directions are not equal.*

1.2 Technical Lemmas

The following result is well known, see for example [21].

Lemma 1.2.1. *Let $D(0,1)$ be the unit disk of $\mathbb{R}^2$. Then, the Laplace operator $\Delta = \partial_x^2 + \partial_y^2 :$ $H^2(D(0,1)) \cap H_0^1(D(0,1)) \longrightarrow L^2(D(0,1))$ is an isomorphism. Moreover, there exists a constant $C > 0$ such that*

$$\|v\|_{H^2(D(0,1))} \leq C \|\Delta v\|_{L^2(D(0,1))}, \forall v \in H^2(D(0,1)) \cap H_0^1(D(0,1)).$$

Here, H^2 and H_0^1 are the usual Sobolev spaces defined, for instance, in Lions-Magenes [21].

In the sequel, assume that $\beta_i,\ i=1,2$ are real numbers such that

$$(-1)^i\beta_i>0,\ i=1,\ 2. \tag{1.3}$$

Lemma 1.2.2. *Assume that $\beta_i,\ i=1,2$ fulfil the condition (1.3). Then, there exists a positive constant K_1 such that for each $(u,v)\in H_\gamma^2(0,1)\times H_\delta^2(0,1)$*

$$\left\|u^{(k)}\right\|_{L^2(0,1)}\leq K_1\left\|u^{(2)}\right\|_{L^2(0,1)},\ k=0,\ 1,$$

$$\left\|v^{(k)}\right\|_{L^2(0,1)}\leq K_1\left\|v^{(2)}\right\|_{L^2(0,1)},\ k=0,\ 1,$$

where

$$H_\gamma^2(0,1)=\left\{u\in H^2(0,1): u'(1)+\beta_2 u(1)=u(0)=0\right\},$$

and

$$H_\delta^2(0,1)=\left\{v\in H^2(0,1): v'(0)+\beta_1 v(0)=v(1)=0\right\}.$$

Proof. Let h_1,h_2 be arbitrary fixed elements of $L^2(0,1)$. Every solution of the ordinary differential equation $u''=h_1$, (respectively, $v''=h_2$,) is of the form

$$u(y) \ = \ \int_0^y\left\{\int_0^x h_1(s)\,ds\right\}dx+yu'(0)+u(0),\ y\in[0,1],$$

(respectively,

$$v(y) \ = \ \int_0^y\left\{\int_0^x h_2(s)\,ds\right\}dx+yv'(0)+v(0),\ y\in[0,1]).$$

The variables $u(0)$ and $u'(0)$ (respectively, $v(0)$ and $v'(0)$) are to be determined in a unique way such that the boundary conditions $u'(1)+\beta_2u(1)=u(0)=0$ (respectively, $v'(0)+\beta_1v(0)=v(1)=0$) are satisfied.

From the preceding representation of the solution (and thus also its derivative) and from the required boundary conditions we obtain the following system to be solved:

$$\begin{cases} (1+\beta_2)\,u'(0)+\beta_2u(0)=-\int_0^1 h_1(s)\,ds-\beta_2\int_0^1\left\{\int_0^x h_1(s)\,ds\right\}dx \\ 0u'(0)+u(0)=0, \end{cases}$$

(respectively,

$$\begin{cases} v'(0)+v(0)=-\int_0^1\left\{\int_0^x h_2(s)\,ds\right\}dx \\ v'(0)+\beta_1v(0)=0). \end{cases}$$

This system in the unknowns $u(0)$ and $u'(0)$ (respectively, $v(0)$ and $v'(0)$) is uniquely solvable if and only if

$$\beta_2 + 1 \neq 0,$$

(respectively,

$$\beta_1 - 1 \neq 0).$$

This condition is verified thanks to (1.3). Finally, the unique solution of the problem

$$\begin{cases} u'' = h_1, \\ u(0) = 0, \\ u'(1) + \beta_2 u(1) = 0, \end{cases}$$

(respectively,

$$\begin{cases} v'' = h_2, \\ v(1) = 0, \\ v'(0) + \beta_1 v(0) = 0), \end{cases}$$

is given by

$$u(y) \quad = \quad \int_0^y \left\{ \int_0^x h_1(s)\, ds \right\} dx + y u'(0),$$

(respectively,

$$v(y) \quad = \quad \int_0^y \left\{ \int_0^x h_2(s)\, ds \right\} dx + y v'(0) + v(0)),$$

where

$$u'(0) = \frac{-\int_0^1 h_1(s)\, ds - \beta_2 \int_0^1 \left\{ \int_0^x h_1(s)\, ds \right\} dx}{\beta_2 + 1},$$

(respectively,

$$\begin{cases} v(0) = \dfrac{-\int_0^1 \left\{ \int_0^x h_2(s)\, ds \right\} dx}{\beta_1 - 1} \\ v'(0) = -\beta_1 v(0)). \end{cases}$$

Using the Cauchy-Schwarz inequality, we obtain the following estimates

$$|u'(0)| \leq C \|h_1\|_{L^2(0,1)},$$

(respectively,

$$|v(0)| \leq C \|h_2\|_{L^2(0,1)}$$
$$|v'(0)| \leq C \|h_2\|_{L^2(0,1)}),$$

which will allow us to obtain the desired estimates. $\square$

Lemma 1.2.3. *Under the assumption (1.3) on $\beta_i,\ i=1,2,$ there exists a positive constant C_1 (independent of a and b) such that for each $(u,v)\in H^2_\gamma(a,b)\times H^2_\delta(a,b)$*

$$\left\|u^{(k)}\right\|^2_{L^2(a,b)}\leq C_1\,(b-a)^{2(2-k)}\left\|u^{(2)}\right\|^2_{L^2(a,b)},\ k=0,\ 1,$$
$$\left\|v^{(k)}\right\|^2_{L^2(a,b)}\leq C_1\,(b-a)^{2(2-k)}\left\|v^{(2)}\right\|^2_{L^2(a,b)},\ k=0,\ 1,$$

where,

$$H^2_\gamma(a,b)=\left\{u\in H^2(a,b): u(a)=0, u'(b)+\frac{\beta_2}{b-a}u(b)=0\right\},$$
$$H^2_\delta(a,b)=\left\{v\in H^2(a,b): v'(a)+\frac{\beta_1}{b-a}v(a)=0, v(b)=0\right\}.$$

Proof. It is a direct consequence of Lemma 1.2.2 by using the following affine change of variable

$$[0,1]\longrightarrow[a,b],\ x\mapsto(1-x)\,a+xb=y.$$

□

1.3 Some model parabolic problems

The following results are consequences of Theorem 4.3 ([21], Vol.2).

Proposition 1.3.1. *Let R be the cylinder $]0,T[\times B(0,1)$ where $B(0,1)$ is the unit disk of $\mathbb{R}^2$, $f\in L^2(R)$ and $u_0\in H^1(\gamma_0)$. Then the problem*

$$\begin{cases}\partial_t u-\Delta u=f\ \text{ in } R,\\ u|_{\gamma_0}=u_0,\\ u|_{\gamma_1}=0,\end{cases}$$

where $\Delta=\sum_{j=1}^{2}\partial^2_{x_j}$, $\gamma_0=\{0\}\times B(0,1)$ et $\gamma_1=\,]0,T[\,\times\partial B(0,1)$, admits a (unique) solution $u\in H^{1,2}(R)$ if and only if the following compatibility condition is satisfied

$$u_0|_{\partial\gamma_0}=0.$$

Proposition 1.3.2. *(Theorem 4.3, [21]) Let $Q=]0,T[\times]0,1[$, $f\in L^2(Q)$ and $\phi\in H^1(\gamma_0)$. Then the following initial/boundary value problems :*

$$\begin{cases}\partial_t u-\partial^2_x u=f_1\in L^2(Q),\\ u|_{\Gamma_0}=\phi,\\ u|_{\Gamma_1}=0,\\ \partial_x u+\beta_2 u|_{\Gamma_2}=0,\end{cases}$$

and

$$\begin{cases} \partial_t v - \partial_x^2 v = f_2 \in L^2(Q), \\ v|_{\Gamma_0} = \phi, \\ v|_{\Gamma_2} = 0, \\ \partial_x v + \beta_1 v|_{\Gamma_1} = 0, \end{cases}$$

admit unique solutions in $\mathcal{H}^{1,2}(Q)$. *Here,* $\gamma_0 = \{0\}\times]0,1[$, $\gamma_1 =]0,T[\times\{0\}$ *and* $\gamma_2 =]0,T[\times\{1\}$,. *The coefficients* $\beta_i, \imath = 1,2$ *are real numbers satisfying* $(-1)^i\beta_i > 0$.

CHAPTER 2

GLOBAL IN TIME RESULTS FOR A PARABOLIC EQUATION SOLUTION IN NON-RECTANGULAR DOMAINS

This chapter deals with the parabolic equation

$$\partial_t w - c(t)\partial_x^2 w = f \text{ in } D,\ D = \left\{(t,x) \in \mathbb{R}^2 : t > 0,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

with $\varphi_i : [0,+\infty[\to \mathbb{R}, i = 1,\ 2$ and $c : [0,+\infty[\to \mathbb{R}$ satisfying some conditions and the problem is supplemented with boundary conditions of Dirichlet-Robin type. We study the global regularity problem in a suitable parabolic Sobolev space. We prove in particular that for $f \in L^2(D)$ there exists a unique solution w such that $w,\ \partial_t w,\ \partial^j w \in L^2(D), j = 1,\ 2$. Notice that the case of bounded non-rectangular domains is studied in [15]. The proof is based on energy estimates after transforming the problem in a strip region combined with some interpolation inequality. This work complements the results obtained in [26] in the case of Cauchy-Dirichlet boundary conditions.

2.1 Introduction and statement of the main result

Let D be an open set of $\mathbb{R}^2$ defined by

$$D := \left\{(t,x) \in \mathbb{R}^2 : t > 0,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

where $\varphi_i \in C([0,+\infty[) \cap C^1(0,+\infty), i = 1,\ 2,$

$$\varphi(t) := \varphi_2(t) - \varphi_1(t) > 0 \quad \forall t > 0, \text{ and } \varphi(0) = 0.$$

The lateral boundaries of D are defined by

$$\Gamma_i = \left\{(t, \varphi_i(t)) \in \mathbb{R}^2 : t > 0\right\},\ i = 1,\ 2.$$

Let us introduce the following functional space:

$$\mathcal{H}^{1,2}(D) := \left\{w \in L^2(D) : \partial_t w, \partial_x w, \partial_x^2 w \in L^2(D)\right\}$$

where $L^2(D)$ stands for the usual Lebesgue space of square-integrable functions on D. The space $\mathcal{H}^{1,2}(D)$ is equipped with the natural norm, that is

$$\|w\|^2_{\mathcal{H}^{1,2}(D)} = \|w\|^2_{L^2(D)} + \|\partial_t w\|^2_{L^2(D)} + \sum_{j=1}^{2} \left\|\partial_x^j w\right\|^2_{L^2(D)}.$$

We consider the problems: to find a function $u \in \mathcal{H}^{1,2}(D)$ (respectively, $v \in \mathcal{H}^{1,2}(D)$) that satisfies the equation

$$\partial_t u - c(t)\partial_x^2 u = f_1 \text{ a.e. on } D \tag{2.1}$$

(respectively,

$$\partial_t v - c(t)\partial_x^2 v = f_2 \text{ a.e. on } D) \tag{2.2}$$

and the boundary conditions

$$u|_{\Gamma_1} = \partial_x u + \beta_2 u|_{\Gamma_2} = 0, \tag{2.3}$$

(respectively,

$$v|_{\Gamma_2} = \partial_x v + \beta_1 v|_{\Gamma_1} = 0), \tag{2.4}$$

where $f_i \in L^2(D),\ i = 1,\ 2$ and the coefficient c is a continuous real-valued function defined on $[0, +\infty[$, differentiable on $]0, +\infty[$ and such that

$$0 < \alpha \leq c(t) \leq \beta$$

for every $t \in [0, +\infty[$, where α and β are positive constants. Here, the coefficient $\beta_i,\ i = 1,\ 2$ in boundary conditions are real numbers such that

$$\beta_1 < 0 \text{ and } \beta_2 > 0.$$

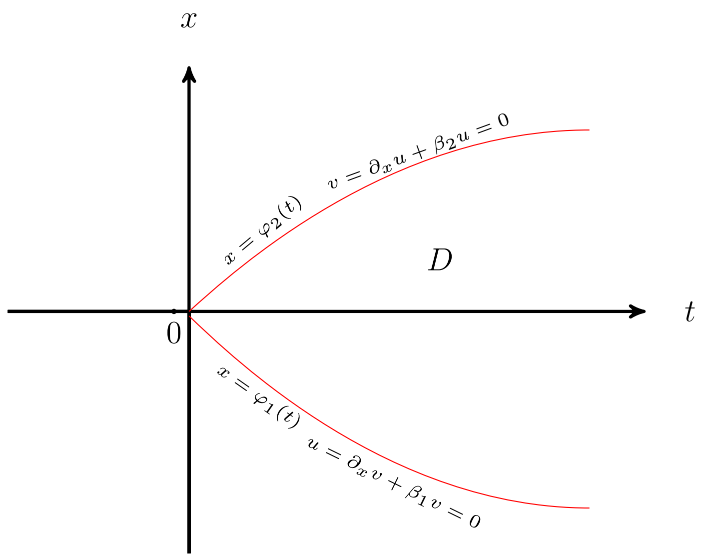

Fig.1: The unbounded non-rectangular domain D.

Problems (2.1)-(2.3) and (2.2)-(2.4) modelize, for instance, the lateral diffusion of two pollutants in a flow of a river with variable width. Note that the Robin type conditions

$$\partial_x v + \beta_1 v|_{\Gamma_1} = \partial_x u + \beta_2 u|_{\Gamma_2} = 0,$$

mean for instance, that the flux of diffusion of the pollutants are proportional to their propagations along the wide of the river. The most interesting points of the parabolic problems studied here is the unboundedness of D with respect to the time variable t and the fact that D shrinks at $t = 0$ ($\varphi(0) = 0$) which prevent one using the methods in [20] and [21]. It is well known that there are two main approaches for the study of boundary value problems in such non-regular domains. The analysis can be done in weighted spaces with the weight controlling the behavior of the solutions near the singularity of the boundary of the domain (see, for instance, [17], [18] and [19]). Our approach is different. Indeed, the space $\mathcal{H}^{1,2}$ used here has low smoothness but one must add assumptions on the type of the domain D, as well as conditions on the coefficients c and β_i, $i = 1,\ 2$, near the singular point 0 and in the neighborhood of $+\infty$. So, our main result is the following:

Theorem 2.1.1. *Let us assume that*

$$|\varphi_i'(t)|\,\varphi(t) \quad \rightarrow \quad 0 \qquad \text{as } t \rightarrow 0^+, \quad i = 1,\ 2, \tag{2.5}$$

$$(-1)^i\,(2c(t)\beta_i - \varphi_i'(t)) \geq 0 \quad \text{a.e. } t \in\,]0, +\infty[\,,\ i = 1,\ 2, \tag{2.6}$$

$$\varphi \text{ and } \varphi' \text{ are uniformly bounded in a neighborhood of} + \infty, \tag{2.7}$$

$$c \text{ is a decreasing function in}]0, +\infty[\,, \tag{2.8}$$

and one of the following conditions is satisfied

(a) φ *is increasing in a neighborhood of* $+\infty$,

(b) $\exists M > 0 : |\varphi'|\,\varphi \leq Mc(t)$.

Then Problem (2.1),(2.3) (respectively, Problem (2.2),(2.4)) admits a unique solution $u \in \mathcal{H}^{1,2}(D)$ *(respectively,* $v \in \mathcal{H}^{1,2}(D)$*).*

The case where D is bounded (with $c(t) = 1$) is studied in [15]. The case where $\beta_1 = \infty$ (or $\beta_2 = \infty$) corresponding to Cauchy-Dirichlet boundary conditions is studied in [26]. Whereas second-order parabolic equations in bounded non-cylindrical domains are well studied (see for instance [2], [7], [11], [22], [23], [25] and the references therein), the literature concerning unbounded non-cylindrical domains does not seem to be very rich. The regularity of the heat equation solution in a non-smooth and unbounded domain (in the x direction) is obtained in [14], [10] and [3].

In the next sections, we prove Theorem 2.1.1 in four steps:

(1) case of a bounded domain which can be transformed into a rectangle;

(2) case of an unbounded domain which can be transformed into a half strip;

(3) case of a small in time bounded triangular domain;

(4) finally, we use the previous steps and a trace result to complete the proof of Theorem 2.1.1.

2.2 The case of a bounded domain which can be transformed into a rectangle

Let T be an arbitrary positive number. Denote by

$$D_1 := \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T;\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

with $\varphi(t) > 0$ for all $t \in [0,T]$ and consider the following problems:

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f_1 \ \text{ a.e. on } D_1, \\ u|_{\Gamma_1} = u|_{\Gamma_0} = 0, \\ \partial_x u + \beta_2 u|_{\Gamma_2} = 0, \end{cases} \tag{2.9}$$

and

$$\begin{cases} \partial_t v - c(t)\partial_x^2 v = f_2 \ \text{ a.e. on } D_1, \\ v|_{\Gamma_2} = v|_{\Gamma_0} = 0, \\ \partial_x v + \beta_1 v|_{\Gamma_1} =, \end{cases} \tag{2.10}$$

where $f_i \in L^2(D_1),\ i = 1,\ 2$ and Γ_0 is the part of ∂D_1 where $t = 0$.

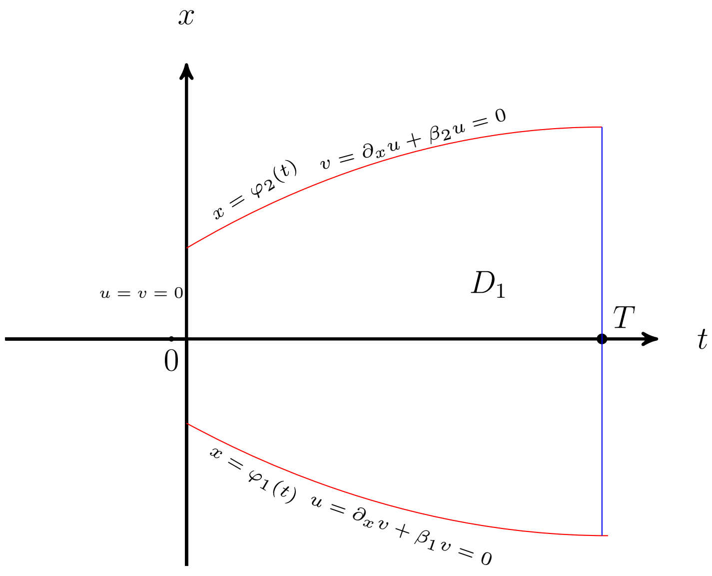

Fig.2: The bounded domain D_1.

Let us denote the inner product in $L^2(D_1)$ by $\langle ., . \rangle$. Then, the uniqueness of the solutions may be obtained by developing the inner products

$$\left\langle \partial_t u - c(t)\partial_x^2 u, u \right\rangle \text{ and } \left\langle \partial_t v - c(t)\partial_x^2 v, v \right\rangle.$$

Indeed, Let us consider $u \in \mathcal{H}^{1,2}(D_1)$ (respectively, $v \in \mathcal{H}^{1,2}(D_1)$) a solution of Problem (2.9) (respectively, of Problem (2.10)) with a null right-hand side terms. So,

$$\partial_t u - c(t)\partial_x^2 u = \partial_t v - c(t)\partial_x^2 v = 0 \text{ in } D_1.$$

In addition u and v fulfil the boundary conditions

$$u|_{\Gamma_0} = v|_{\Gamma_0} = u|_{\Gamma_1} = v|_{\Gamma_2} = \partial_x v + \beta_1 v|_{\Gamma_1} = \partial_x u + \beta_2 u|_{\Gamma_2} = 0.$$

Using Green formula, we have

$$\begin{aligned}
&\textstyle\int_{D_1} \left(\partial_t u - c(t)\partial_x^2 u\right) u \; dt \; dx + \int_{D_1} \left(\partial_t v - c(t)\partial_x^2 v\right) v \; dt \; dx \\
&= \textstyle\int_{\partial D_1} \left(\frac{1}{2}|u|^2 \nu_t - c(t)\partial_x u.u\nu_x\right) d\sigma + \int_{\partial D_1} \left(\frac{1}{2}|v|^2 \nu_t - c(t)\partial_x v.v\nu_x\right) d\sigma \\
&+ \textstyle\int_{D_1} c(t)\left(|\partial_x u|^2\right) dt \; dx + \int_{D_1} c(t)\left(|\partial_x v|^2\right) dt \; dx,
\end{aligned}$$

where ν_t, ν_x are the components of the unit outward normal vector at ∂D_1. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of D_1 where $t = 0$, we have $u = v = 0$. Accordingly the corresponding boundary integrals vanish. On the part of the boundary of D_1 where $t = T$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\frac{1}{2}\int_{\varphi_1(T)}^{\varphi_2(T)} [|u|^2 (T,x) + |v|^2 (T,x)]dx,$$

is nonnegative. On the parts of the boundary where $x = \varphi_i(t)$, $i = 1, 2$, we have

$$\nu_x = \frac{(-1)^i}{\sqrt{1 + (\varphi_i')^2 (t)}}, \ \nu_t = \frac{(-1)^{i+1} \varphi_i'(t)}{\sqrt{1 + (\varphi_i')^2 (t)}}$$

and

$$u(t,\varphi_1(t)) = v(t,\varphi_2(t)) = \partial_x u(t,\varphi_2(t)) + \beta_2 u(t,\varphi_2(t)) = \partial_x v(t,\varphi_1(t)) + \beta_1 v(t,\varphi_1(t)) = 0.$$

Consequently, the corresponding integral is

$$\int_0^T (2c(t)\beta_2 - \varphi_2'(t))\, u^2(t,\varphi_2(t))\, dt + \int_0^T (-2c(t)\beta_1 + \varphi_1'(t))\, v^2(t,\varphi_1(t))\, dt.$$

Then, we obtain

$$\begin{aligned}
&\textstyle\int_{D_1} (\partial_t u - c(t)\partial_x^2 u)\, u \ dt \ dx + \int_{D_1} (\partial_t v - c(t)\partial_x^2 v)\, v \ dt \ dx \\
= \ & \int_0^T (2c(t)\beta_2 - \varphi_2'(t))\, u^2(t,\varphi_2(t))\, dt + \int_0^T (-2c(t)\beta_1 + \varphi_1'(t))\, v^2(t,\varphi_1(t))\, dt \\
&+\frac{1}{2}\int_{\varphi_1(T)}^{\varphi_2(T)} [|u|^2 (T,x) + |v|^2 (T,x)]dx + \int_{D_1} c(t)\left(|\partial_x u|^2\right) dt \ dx + \int_{D_1} c(t)\left(|\partial_x v|^2\right) dt \ dx.
\end{aligned}$$

Consequently using the fact that u and v are the solutions yields

$$\int_{D_1} c(t)\left(|\partial_x u|^2\right) dt \ dx + \int_{D_1} c(t)\left(|\partial_x v|^2\right) dt \ dx = 0,$$

because thanks to the condition (2.6) and to the fact that $c(t) > 0$ for every $t \in [0, +\infty[$, we have

$$\begin{aligned}
&\int_0^T (2c(t)\beta_2 - \varphi_2'(t))\, u^2(t,\varphi_2(t))\, dt + \int_0^T (-2c(t)\beta_1 + \varphi_1'(t))\, v^2(t,\varphi_1(t))\, dt \\
&+\frac{1}{2}\int_{\varphi_1(T)}^{\varphi_2(T)} [|u|^2 (T,x) + |v|^2 (T,x)]dx + \int_{D_1} c(t)\left(|\partial_x u|^2\right) dt \ dx \geq 0.
\end{aligned}$$

This implies that $|\partial_x u|^2 + |\partial_x v|^2 = 0$ and consequently $\partial_x^2 u = \partial_x^2 v = 0$. Then, the hypothesis $\partial_t u - c(t)\partial_x^2 u = \partial_t v - c(t)\partial_x^2 v = 0$ gives $\partial_t u = \partial_t v = 0$. Thus, u, v are constants. The boundary conditions and the fact that $\beta_i \neq 0$, $i = 1, 2$ imply that $u = v = 0$ in D_1. This proves the uniqueness of the solutions of Problems (2.9) and (2.10).

Now, let us look at the existence of solutions for Problems (2.9) and (2.10). The change of variables (t, x) to $\left(t, \frac{x-\varphi_1(t)}{\varphi(t)}\right)$ transforms D_1 into the rectangle $Q =]0, T[\times]0, 1[$ and Problem (2.9) (respectively, Problem (2.10)) becomes the following:

$$\begin{cases} \partial_t u + a\,(t, x)\,\partial_x u - \frac{c(t)}{\varphi^2(t)}\partial_x^2 u = f_1 \text{ a.e. on } Q, \\ u|_{t=0} = u|_{x=0} = 0, \\ \partial_x u + \beta_2 \varphi(t) u|_{x=1} = 0, \end{cases}$$

(respectively,

$$\begin{cases} \partial_t v + a\,(t, x)\,\partial_x v - \frac{c(t)}{\varphi^2(t)}\partial_x^2 v = f_2 \text{ a.e. on } Q, \\ v|_{t=0} = v|_{x=1} = 0, \\ \partial_x v + \beta_1 \varphi(t) v|_{x=0} = 0), \end{cases}$$

where $f_i \in L^2(Q)$, $i = 1, 2$ and $a\,(t, x) = -\frac{x\varphi'(t)+\varphi_1'(t)}{\varphi(t)}$. Observe that the coefficient a is bounded. So, the operator

$$a(t, x)\partial_x : \mathcal{H}^{1,2}(Q) \longrightarrow L^2(Q)$$

is compact. Hence, it is sufficient to study the following problem:

$$\begin{cases} \partial_t u - \frac{c(t)}{\varphi^2(t)}\partial_x^2 u = f_1 \text{ a.e. on } Q, \\ u|_{t=0} = u|_{x=0} = 0, \\ \partial_x u + \beta_2 \varphi(t) u|_{x=1} = 0, \end{cases} \tag{2.11}$$

(respectively,

$$\begin{cases} \partial_t v - \frac{c(t)}{\varphi^2(t)}\partial_x^2 v = f_2 \text{ a.e. on } Q, \\ v|_{t=0} = v|_{x=1} = 0, \\ \partial_x v + \beta_1 \varphi(t) v|_{x=0} = 0), \end{cases} \tag{2.12}$$

where $f_i \in L^2(Q),\ i = 1,\ 2$. It is clear that Problem (2.11) (respectively, Problem (2.12)) admits a (unique) solution $u \in \mathcal{H}^{1,2}(Q)$ (respectively, $v \in \mathcal{H}^{1,2}(Q)$) because the coefficient $\frac{c(t)}{\varphi^2(t)}$ satisfies the "uniform parabolicity" condition (see, for example [1]). On other hand, it is easy to verify that the aforementioned change of variable conserves the spaces L^2 and $\mathcal{H}^{1,2}$. Consequently, we have proved the following theorem:

Theorem 2.2.1. *Problem (2.9) (respectively, Problem (2.10)) admits a (unique) solution $u \in \mathcal{H}^{1,2}(D_1)$ (respectively, $v \in \mathcal{H}^{1,2}(D_1)$).*

2.3 The case of an unbounded domain which can be transformed into a half strip

In this case, we set

$$D_2 := \left\{(t,x) \in \mathbb{R}^2 : t > 0;\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

with $\varphi(0) > 0$ and consider the following problems:

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f_1 \ \text{ a.e. on } D_2, \\ u|_{\Gamma_1} = u|_{\Gamma_0} = 0, \\ \partial_x u + \beta_2 u|_{\Gamma_2} = 0, \end{cases} \tag{2.13}$$

and

$$\begin{cases} \partial_t v - c(t)\partial_x^2 v = f_2 \ \text{ a.e. on } D_2, \\ v|_{\Gamma_2} = v|_{\Gamma_0} = 0, \\ \partial_x v + \beta_1 v|_{\Gamma_1} = 0, \end{cases} \tag{2.14}$$

where $f_i \in L^2(D_2),\ i = 1,\ 2$ and Γ_0 is the part of ∂D_2 where $t = 0$.

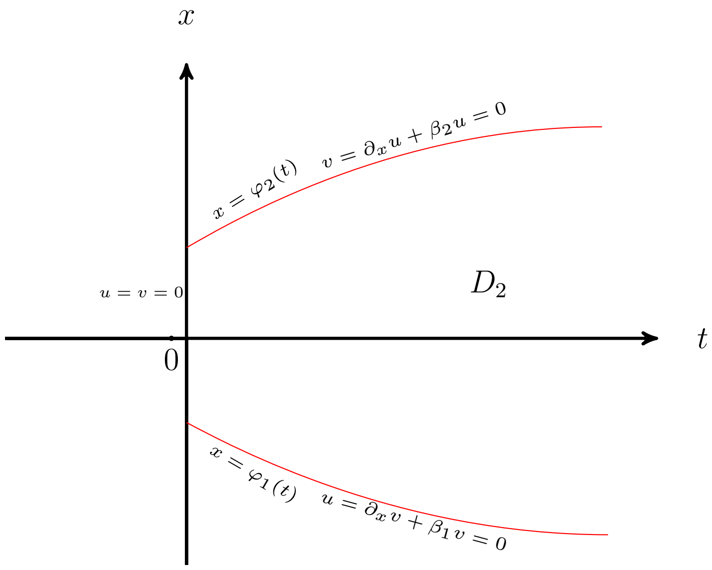

Fig.3: The unbounded domain D_2.

The change of variables indicated in the previous section transforms D_2 into the half strip $P =]0, +\infty[\times]0, 1[$. So Problem (2.13) (respectively, Problem (2.14)) can be written as follows:

$$\begin{cases} \partial_t u + a\,(t,x)\,\partial_x u - \frac{c(t)}{\varphi^2(t)}\partial_x^2 u = f_1 \text{ a.e. on } P, \\ u|_{t=0} = u|_{x=0} = 0, \\ \partial_x u + \beta_2 \varphi(t) u|_{x=1} = 0, \end{cases} \tag{2.15}$$

(respectively,

$$\begin{cases} \partial_t v + a\,(t,x)\,\partial_x v - \frac{c(t)}{\varphi^2(t)}\partial_x^2 v = f_2 \text{ a.e. on } P, \\ v|_{t=0} = v|_{x=1} = 0, \\ \partial_x v + \beta_1 \varphi(t) v|_{x=0} = 0), \end{cases} \tag{2.16}$$

where $f_i \in L^2(P),\ i = 1,\ 2$ and the coefficients a is that defined in Section 2.2.

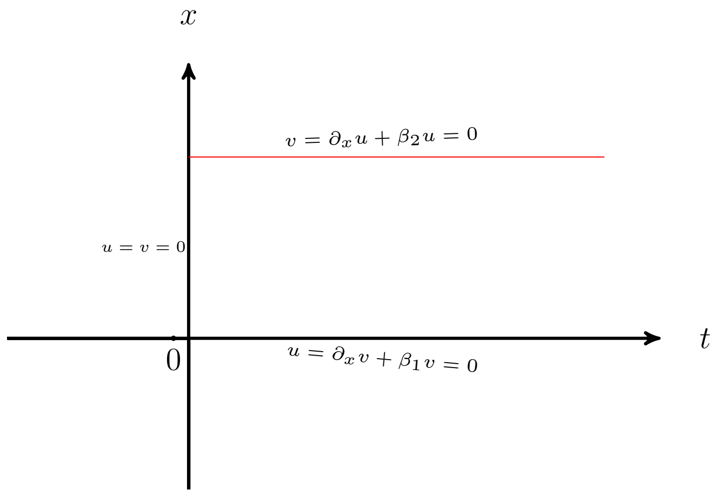

Fig.4: The half strip P.

Let $f_i^{(n)},\ i=1,\ 2$ be the restriction $f_i|_{]0,n[\times]0,1[}\ ,\ i=1,\ 2$ for all $n \in \mathbb{N}^*$. Then, Theorem 2.2.1 shows that for all $n \in \mathbb{N}^*$, there exists a function $u_n \in \mathcal{H}^{1,2}(P_n)$ (respectively, $v_n \in \mathcal{H}^{1,2}(P_n)$) which solves the problem

$$\begin{cases} \partial_t u_n + a(t,x)\,\partial_x u_n - \frac{c(t)}{\varphi^2(t)}\partial_x^2 u_n = f_1^{(n)} \ \text{ a.e. on } P_n, \\ u_n|_{t=0} = u_n|_{x=0} = 0, \\ \partial_x u_n + \beta_2\varphi(t)u_n|_{x=1} = 0, \end{cases} \tag{2.17}$$

(respectively,

$$\begin{cases} \partial_t v_n + a(t,x)\,\partial_x v_n - \frac{c(t)}{\varphi^2(t)}\partial_x^2 v_n = f_2^{(n)} \ \text{ a.e. on } P_n, \\ v_n|_{t=0} = v_n|_{x=1} = 0, \\ \partial_x v_n + \beta_1\varphi(t)v_n|_{x=0} = 0), \end{cases} \tag{2.18}$$

where $f_i^{(n)} \in L^2(P_n),\ i=1,\ 2$ and $P_n = \,]0,n[\,\times\,]0,1[\,$.

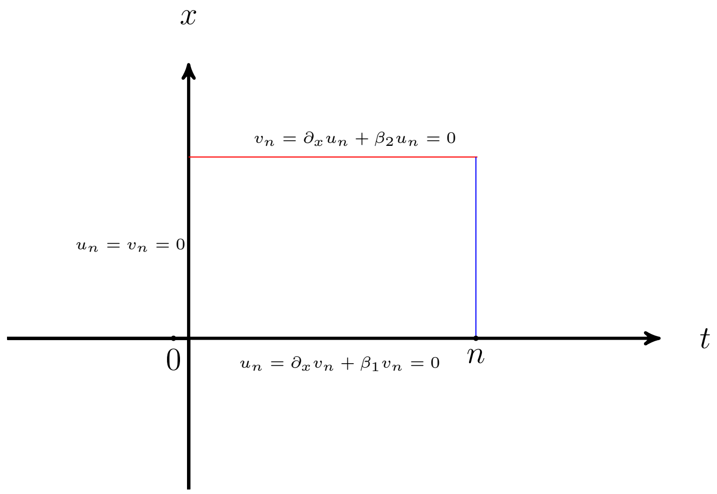

Fig.5: The truncated half strip P_n.

Now, let us prove an "energy" type estimate for the solutions u_n (respectively, v_n) which will allow us to solve Problem (2.15) (respectively, Problem (2.16)) and then equivalently Problem (2.13) (respectively, Problem (2.14)).

Proposition 2.3.1. *There exists a constant $K > 0$ independent of n such that*

$$\|u_n\|^2_{\mathcal{H}^{1,2}(P_n)} \leq K \left\| f_1^{(n)} \right\|^2_{L^2(P_n)} \leq K \|f_1\|^2_{L^2(P)},$$

$$\|v_n\|^2_{\mathcal{H}^{1,2}(P_n)} \leq K \left\| f_2^{(n)} \right\|^2_{L^2(P_n)} \leq K \|f_2\|^2_{L^2(P)}.$$

In order to prove Proposition 2.3.1, we need the following result:

Lemma 2.3.1. *There exists a constant K independent of n such that*

$$\|u_n\|^2_{L^2(P_n)} \leq K \|\partial_x u_n\|^2_{L^2(P_n)} \leq K \|f_1\|^2_{L^2(P)},$$

$$\|v_n\|^2_{L^2(P_n)} \leq K \|\partial_x v_n\|^2_{L^2(P_n)} \leq K \|f_2\|^2_{L^2(P)}.$$

Proof. The Poincaré inequality gives $\|u_n\|_{L^2(P_n)} \leq K \|\partial_x u_n\|_{L^2(P_n)}$ and $\|v_n\|_{L^2(P_n)} \leq K \|\partial_x v_n\|_{L^2(P_n)}$. Now, we estimate the inner products $\left\langle f_1^{(n)}, u_n \right\rangle$ and $\left\langle f_2^{(n)}, v_n \right\rangle$ in $L^2(P_n)$.

1) Estimation of $\left\langle f_1^{(n)}, u_n \right\rangle$:

$$\begin{array}{rcl} \left\langle f_1^{(n)}, u_n \right\rangle & = & \int_{P_n} u_n \partial_t u_n dtdx + \int_{P_n} a(t,x) u_n \partial_x u_n dtdx - \int_{P_n} \frac{c(t)}{\varphi^2(t)} u_n \partial_x^2 u_n dtdx \\ & = & \int_{\partial P_n} \left[\frac{1}{2} |u_n|^2 \nu_t + a(t,x) \frac{1}{2} |u_n|^2 \nu_x - \frac{c(t)}{\varphi^2(t)} \partial_x u_n . u_n \nu_x \right] d\sigma \\ & & + \int_{P_n} \frac{c(t)}{\varphi^2(t)} (\partial_x u_n)^2 dtdx - \frac{1}{2} \int_{P_n} \partial_x a(t,x) |u_n|^2 dtdx, \end{array}$$

where ν_t, ν_x are the components of the unit outward normal vector at the boundary of P_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of P_n where $t = 0$, we have $u_n = 0$ and consequently the corresponding boundary integral vanishes. On the part of the boundary where $t = n$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral is the following:

$$\int_0^1 \frac{1}{2} (u_n)^2 (n, x) dx.$$

On the part of the boundary where $x = 0$, we have $\nu_x = -1,\ \nu_t = 0$ and $u_n(t, 0) = 0$. Consequently, the corresponding integral vanishes. On the part of the boundary where $x = 1$, we have $\nu_x = 1,\ \nu_t = 0$ and

$$\partial_x u_n (t, 1) + \beta_2 \varphi (t) u_n (t, 1) = 0.$$

Consequently, the corresponding integral is

$$\int_0^n \frac{(2c(t)\beta_2 - \varphi_2' (t))}{2\varphi (t)} (u_n)^2 (t, 1) dt.$$

Finally,

$$\begin{array}{rcl} \left\langle f_1^{(n)}, u_n \right\rangle & = & \int_{\varphi_1(n)}^{\varphi_2(n)} \frac{1}{2} (u_n)^2 (n, x) dx + \int_0^n \frac{(2c(t)\beta_2 - \varphi_2'(t))}{2\varphi(t)} (u_n)^2 (t, 1) dt + \int_{P_n} \frac{c(t)}{\varphi^2(t)} (\partial_x u_n)^2 dt dx \\ & & + \frac{1}{2} \int_{P_n} \frac{\varphi'(t)}{\varphi(t)} |u_n|^2 dt dx. \end{array}$$

Thanks to the condition (2.6) and since the function φ increases, we obtain

$$\left\langle f_1^{(n)}, u_n \right\rangle \geq \int_{P_n} \frac{c(t)}{\varphi^2 (t)} (\partial_x u_n)^2 dt dx \geq C \|\partial_x u_n\|^2_{L^2(P_n)}.$$

Hence, for all $\epsilon > 0$,

$$\begin{array}{rcl} \|\partial_x u_n\|^2_{L^2(P_n)} & \leq & \frac{1}{C} \|u_n\|_{L^2(P_n)} \left\| f_1^{(n)} \right\|_{L^2(P_n)} \\ & \leq & \frac{1}{C\epsilon} \left\| f_1^{(n)} \right\|^2_{L^2(P_n)} + \frac{\epsilon}{C} \|u_n\|^2_{L^2(P_n)}. \end{array}$$

By using the Poincaré inequality, we obtain

$$(1 - \frac{\epsilon}{C}) \|\partial_x u_n\|^2_{L^2(P_n)} \leq \frac{1}{C\epsilon} \left\| f_1^{(n)} \right\|^2_{L^2(P_n)}.$$

Choosing ϵ small enough in the previous inequality, we prove the existence of a constant K such that

$$\|\partial_x u_n\|^2_{L^2(P_n)} \leq K \left\| f_1^{(n)} \right\|^2_{L^2(P_n)}.$$

Since

$$\left\| f_1^{(n)} \right\|^2_{L^2(P_n)} \leq \| f_1 \|^2_{L^2(P)} ,$$

we obtain

$$\| \partial_x u_n \|^2_{L^2(P_n)} \leq K \| f_1 \|^2_{L^2(P)} .$$

1) Estimation of $\left\langle f_2^{(n)}, v_n \right\rangle$: We have

$$\begin{aligned} \left\langle f_2^{(n)}, v_n \right\rangle &= \int_{P_n} v_n \partial_t v_n dt dx + \int_{P_n} a\left(t,x\right) v_n \partial_x v_n dt dx - \int_{P_n} \tfrac{c(t)}{\varphi^2(t)} v_n \partial_x^2 v_n dt dx \\ &= \int_{\partial P_n} \left[\tfrac{1}{2} \left| v_n \right|^2 \nu_t + a(t,x) \tfrac{1}{2} \left| v_n \right|^2 \nu_x - \tfrac{c(t)}{\varphi^2(t)} \partial_x v_n . v_n \nu_x \right] d\sigma \\ & \int_{P_n} + \tfrac{c(t)}{\varphi^2(t)} \left(\partial_x v_n \right)^2 dt dx - \tfrac{1}{2} \int_{P_n} \partial_x a(t,x) \left| v_n \right|^2 dt dx, \end{aligned}$$

where ν_t, ν_x are the components of the unit outward normal vector at the boundary of P_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of P_n where $t = 0$, we have $v_n = 0$ and consequently the corresponding boundary integral vanishes. On the part of the boundary where $t = n$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral is the following:

$$\int_0^1 \frac{1}{2} \left(v_n \right)^2 (n,x) dx.$$

On the part of the boundary where $x = 0$, we have $\nu_x = -1,\ \nu_t = 0$ and

$$\partial_x v_n \left(t, 0\right) + \beta_1 \varphi\left(t\right) v_n \left(t, 0\right) = 0.$$

Consequently, the corresponding integral is

$$\int_0^n \frac{\left(-2c(t)\beta_1 + \varphi_1'\left(t\right)\right)}{2\varphi\left(t\right)} \left(v_n\right)^2 (t,0) dt.$$

On the part of the boundary where $x = 1$, we have $\nu_x = 1,\ \nu_t = 0$ and $v_n \left(t, 1\right)) = 0$. Consequently, the corresponding integral vanishes. Finally,

$$\begin{aligned} \left\langle f_2^{(n)}, v_n \right\rangle &= \int_{\varphi_1(n)}^{\varphi_2(n)} \tfrac{1}{2} \left(v_n \right)^2 (n,x) dx + \int_0^n \tfrac{\left(-2c(t)\beta_1 + \varphi_1'(t)\right)}{2\varphi(t)} \left(v_n \right)^2 (t,0) dt + \int_{P_n} \tfrac{c(t)}{\varphi^2(t)} \left(\partial_x v_n \right)^2 dt dx \\ & + \tfrac{1}{2} \int_{P_n} \tfrac{\varphi'(t)}{\varphi(t)} \left| v_n \right|^2 dt dx, \end{aligned}$$

Thanks to the condition (2.6) and since the function φ increases, we obtain

$$\left\langle f_2^{(n)}, v_n \right\rangle \geq \int_{P_n} \frac{c(t)}{\varphi^2\left(t\right)} \left(\partial_x v_n \right)^2 dt dx \geq C \left\| \partial_x v_n \right\|^2_{L^2(P_n)} .$$

Hence, for all $\epsilon > 0$,

$$\begin{aligned} \| \partial_x v_n \|^2_{L^2(P_n)} &\leq \tfrac{1}{C} \| v_n \|_{L^2(P_n)} \left\| f_2^{(n)} \right\|_{L^2(P_n)} \\ &\leq \tfrac{1}{C\epsilon} \left\| f_2^{(n)} \right\|^2_{L^2(P_n)} + \tfrac{\epsilon}{C} \| v_n \|^2_{L^2(P_n)} . \end{aligned}$$

By using the Poincaré inequality, we obtain

$$(1-\frac{\epsilon}{C})\left\|\partial_x v_n\right\|^2_{L^2(P_n)} \leq \frac{1}{C\epsilon}\left\|f_2^{(n)}\right\|^2_{L^2(P_n)}.$$

Choosing ϵ small enough in the previous inequality, we prove the existence of a constant K such that

$$\left\|\partial_x v_n\right\|^2_{L^2(P_n)} \leq K\left\|f_2^{(n)}\right\|^2_{L^2(P_n)}.$$

Since

$$\left\|f_2^{(n)}\right\|^2_{L^2(P_n)} \leq \left\|f_2\right\|^2_{L^2(P)},$$

we obtain

$$\left\|\partial_x v_n\right\|^2_{L^2(P_n)} \leq K\left\|f_2\right\|^2_{L^2(P)}.$$

$\square$

Remark 2.3.1. *Similar computations show that the same result holds true when we substitute the condition that φ increases in a neighborhood of $+\infty$ by the following:*

$$\left|\varphi'(t)\right|\varphi(t) \leq Mc(t).$$

Proof of Proposition 2.3.1

Let us denote the inner product in $L^2(P_n)$ by $\langle .,.\rangle$, and set $L := \partial_t + a(t,x)\,\partial_x - \frac{c(t)}{\varphi^2(t)}\partial_x^2$, then we have

$$\begin{aligned}
\left\|f_1^{(n)}\right\|^2_{L^2(P_n)} &= \langle \partial_t u_n + a(t,x)\,\partial_x u_n - \tfrac{c(t)}{\varphi^2(t)}\partial_x^2 u_n, \partial_t u_n + a(t,x)\,\partial_x u_n - \tfrac{c(t)}{\varphi^2(t)}\partial_x^2 u_n\rangle \\
&= \left\|\partial_t u_n\right\|^2_{L^2(P_n)} + \left\|a\partial_x u_n\right\|^2_{L^2(P_n)} + \left\|\tfrac{c(t)}{\varphi^2(t)}\partial_x^2 u_n\right\|^2_{L^2(P_n)} + 2\textstyle\int_{P_n} a\partial_t u_n \partial_x u_n dtdx \\
&\quad -2\textstyle\int_{P_n} a\tfrac{c(t)}{\varphi^2(t)}\partial_x u_n \partial_x^2 u_n dtdx - 2\textstyle\int_{P_n}\tfrac{c(t)}{\varphi^2(t)}\partial_t u_n \partial_x^2 u_n dtdx
\end{aligned}$$

and

$$\begin{aligned}
\left\|f_2^{(n)}\right\|^2_{L^2(P_n)} &= \langle \partial_t v_n + a(t,x)\,\partial_x v_n - \tfrac{c(t)}{\varphi^2(t)}\partial_x^2 v_n, \partial_t u_n + a(t,x)\,\partial_x v_n - \tfrac{c(t)}{\varphi^2(t)}\partial_x^2 v_n\rangle \\
&= \left\|\partial_t v_n\right\|^2_{L^2(P_n)} + \left\|a\partial_x v_n\right\|^2_{L^2(P_n)} + \left\|\tfrac{c(t)}{\varphi^2(t)}\partial_x^2 v_n\right\|^2_{L^2(P_n)} + 2\textstyle\int_{P_n} a\partial_t v_n \partial_x v_n dtdx \\
&\quad -2\textstyle\int_{P_n} a\tfrac{c(t)}{\varphi^2(t)}\partial_x v_n \partial_x^2 v_n dtdx - 2\textstyle\int_{P_n}\tfrac{c(t)}{\varphi^2(t)}\partial_t v_n \partial_x^2 v_n dtdx
\end{aligned}$$

Observe that the coefficients a and $\frac{c(t)}{\varphi^2(t)}$ are bounded. So, thanks to Lemma 3.3.2, for all $\epsilon > 0$ we obtain

$$\begin{aligned}
&\|\partial_t u_n\|^2_{L^2(P_n)} + \left\|\tfrac{c(t)}{\varphi^2(t)}\partial_x^2 u_n\right\|^2_{L^2(P_n)} - 2\int_{P_n}\tfrac{c(t)}{\varphi^2(t)}\partial_t u_n \partial_x^2 u_n dtdx \\
\leq\ & \left\|f_1^{(n)}\right\|^2_{L^2(P_n)} + \|a\partial_x u_n\|^2_{L^2(P_n)} + 2\|\partial_t u_n\|_{L^2(P_n)}\|a\partial_x u_n\|_{L^2(P_n)} + 2\|\partial_x^2 u_n\|_{L^2(P_n)}\left\|a\tfrac{c(t)}{\varphi^2(t)}\partial_x u_n\right\|_{L^2(P_n)} \\
\leq\ & \left\|f_1^{(n)}\right\|^2_{L^2(P_n)} + K_1\left(1+\tfrac{2}{\epsilon}\right)\|\partial_x u_n\|^2_{L^2(P_n)} + \epsilon\|\partial_t u_n\|^2_{L^2(P_n)} + \epsilon\|\partial_x^2 u_n\|_{L^2(P_n)} \\
\leq\ & K_\epsilon\left\|f_1^{(n)}\right\|^2_{L^2(P_n)} + \epsilon\|\partial_t u_n\|^2_{L^2(P_n)} + \epsilon\|\partial_x^2 u_n\|_{L^2(P_n)}
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_t v_n\|^2_{L^2(P_n)} + \left\|\tfrac{c(t)}{\varphi^2(t)}\partial_x^2 v_n)\right\|^2_{L^2(P_n)} - 2\int_{P_n}\tfrac{c(t)}{\varphi^2(t)}\partial_t v_n \partial_x^2 v_n dtdx \\
\leq\ & \left\|f_2^{(n)}\right\|^2_{L^2(P_n)} + \|a\partial_x v_n\|^2_{L^2(P_n)} + 2\|\partial_t v_n\|_{L^2(P_n)}\|a\partial_x v_n\|_{L^2(P_n)} + 2\|\partial_x^2 v_n\|_{L^2(P_n)}\left\|a\tfrac{c(t)}{\varphi^2(t)}\partial_x v_n\right\|_{L^2(P_n)} \\
\leq\ & \left\|f_2^{(n)}\right\|^2_{L^2(P_n)} + K_1\left(1+\tfrac{2}{\epsilon}\right)\|\partial_x v_n\|^2_{L^2(P_n)} + \epsilon\|\partial_t v_n\|^2_{L^2(P_n)} + \epsilon\|\partial_x^2 v_n\|_{L^2(P_n)} \\
\leq\ & K_\epsilon\left\|f_2^{(n)}\right\|^2_{L^2(P_n)} + \epsilon\|\partial_t v_n\|^2_{L^2(P_n)} + \epsilon\|\partial_x^2 v_n\|^2_{L^2(P_n)},
\end{aligned}$$

where K_ϵ and K_1 are constants independent of n. Consequently

$$(1-\epsilon)\left(\|\partial_t u_n\|^2_{L^2(P_n)} + \|\partial_x^2 u_n\|^2_{L^2(P_n)}\right) \leq 2\int_{P_n}\tfrac{c(t)}{\varphi^2(t)}\partial_t u_n\partial_x^2 u_n dtdx + K_\epsilon\left\|f_1^{(n)}\right\|^2_{L^2(P_n)} \tag{2.19}$$

and

$$(1-\epsilon)\left(\|\partial_t v_n\|^2_{L^2(P_n)} + \|\partial_x^2 v_n\|^2_{L^2(P_n)}\right) \leq 2\int_{P_n}\tfrac{c(t)}{\varphi^2(t)}\partial_t v_n\partial_x^2 v_n dtdx + K_\epsilon\left\|f_2^{(n)}\right\|^2_{L^2(P_n)}. \tag{2.20}$$

Estimation of $2\int_{P_n}\frac{c(t)}{\varphi^2(t)}\partial_t u_n\partial_x^2 u_n dtdx$: We have

$$\partial_t u_n\partial_x^2 u_n = \partial_x\left(\partial_t u_n\partial_x u_n\right) - \tfrac{1}{2}\partial_t\left(\partial_x u_n\right)^2.$$

Then

$$\begin{aligned}
2\int_{P_n}\tfrac{c(t)}{\varphi^2(t)}\partial_t u_n\partial_x^2 u_n dtdx &= 2\int_{P_n}\tfrac{c(t)}{\varphi^2(t)}\partial_x\left(\partial_t u_n\partial_x u_n\right)dtdx - \int_{P_n}\tfrac{c(t)}{\varphi^2(t)}\partial_t\left(\partial_x u_n\right)^2 dtdx \\
&= \int_{\partial P_n}\tfrac{c(t)}{\varphi^2(t)}\left[-\left(\partial_x u_n\right)^2\nu_t + 2\partial_t u_n\partial_x u_n\nu_x\right]d\sigma + \int_{P_n}\left(\tfrac{c(t)}{\varphi^2(t)}\right)'(\partial_x u_n)^2 dtdx
\end{aligned}$$

where ν_t, ν_x are the components of the outward normal vector at the boundary of P_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of P_n where $t = 0$, we have $u_n = 0$ and consequently $\partial_x u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = n$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$-\int_0^1 \frac{c(n)}{\varphi^2(n)} (\partial_x u_n)^2 (n, x) dx$$

is negative. On the part of the boundary where $x = 0$, we have $\nu_x = -1,\ \nu_t = 0$ and $u_n(t, 0) = 0$. Consequently, the corresponding integral vanishes. On the part of the boundary where $x = 1$, we have $\nu_x = 1,\ \nu_t = 0$ and

$$\partial_x u_n(t, 1) + \beta_2 \varphi(t) u_n(t, 1) = 0.$$

Consequently, the corresponding integral is

$$\int_0^n \frac{-2\beta_2 c(t)}{\varphi(t)}) \partial_t u_n(t, 1) u_n(t, 1) dt = \frac{-\beta_2 c(n)}{\varphi(n)} u_n^2(n, 1) + \int_0^n \beta_2 (\frac{c(t)}{\varphi(t)})' u_n^2(t, 1) dt,$$

which is negative thanks to the condition (2.8) and to the fact that $\beta_2 > 0$. Finally,

$$\begin{aligned} 2\int_{P_n} \tfrac{c(t)}{\varphi^2(t)} \partial_t u_n \partial_x^2 u_n dt dx &= -\int_0^1 \tfrac{c(n)}{\varphi^2(n)} (\partial_x u_n)^2 (n, x) dx - \tfrac{\beta_2 c(n)}{\varphi(n)} u_n^2(n, 1) + \int_0^n \beta_2 (\tfrac{c(t)}{\varphi(t)})' u_n^2(t, 1) dt \\ &\quad + \int_{P_n} (\tfrac{c(t)}{\varphi^2(t)})' (\partial_x u_n)^2 dt dx \end{aligned}$$

Note that the functions $\frac{c(t)}{\varphi^2(t)}$ and $(\frac{c(t)}{\varphi^2(t)})'$ are bounded. So, by using Lemma 2.3.1, we deduce

$$\begin{aligned} 2\int_{P_n} \tfrac{c(t)}{\varphi^2(t)} \partial_t u_n \partial_x^2 u_n dt dx &\leq \int_{P_n} (\tfrac{c(t)}{\varphi^2(t)})' (\partial_x u_n)^2 dt dx \\ &\leq K_2 \|\partial_x u_n\|^2_{L^2(P_n)} \\ &\leq K_3 \|f_1\|^2_{L^2(P)}, \end{aligned}$$

where K_2 and K_3 are constants independent of n. Consequently, Choosing $\epsilon = \frac{1}{2}$ in the relationship (2.19), we obtain

$$\|\partial_t u_n\|^2_{L^2(P_n)} + \left\|\partial_x^2 u_n\right\|^2_{L^2(P_n)} \leq K \|f_1\|^2_{L^2(P)}.$$

Estimation of $2\int_{P_n} \frac{c(t)}{\varphi^2(t)} \partial_t v_n \partial_x^2 v_n dtdx$: We have

$$\partial_t v_n \partial_x^2 v_n = \partial_x \left(\partial_t v_n \partial_x v_n\right) - \tfrac{1}{2}\partial_t \left(\partial_x v_n\right)^2 .$$

Then

$$\begin{aligned} 2\int_{P_n} \tfrac{c(t)}{\varphi^2(t)} \partial_t v_n \partial_x^2 v_n dtdx &= 2\int_{P_n} \tfrac{c(t)}{\varphi^2(t)} \partial_x \left(\partial_t v_n \partial_x v_n\right) dtdx - \int_{P_n} \tfrac{c(t)}{\varphi^2(t)} \partial_t \left(\partial_x v_n\right)^2 dtdx \\ &= \int_{\partial P_n} \tfrac{c(t)}{\varphi^2(t)} \left[- \left(\partial_x v_n\right)^2 \nu_t + 2\partial_t v_n \partial_x v_n \nu_x \right] d\sigma + \int_{P_n} (\tfrac{c(t)}{\varphi^2(t)})'(\partial_x v_n)^2 dtdx \end{aligned}$$

where ν_t, ν_x are the components of the outward normal vector at the boundary of P_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of P_n where $t = 0$, we have $v_n = 0$ and consequently $\partial_x v_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = n$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$-\int_0^1 \frac{c(n)}{\varphi^2(n)} (\partial_x v_n)^2 (n, x) dx.$$

is negative. On the part of the boundary where $x = 0$, we have $\nu_x = -1,\ \nu_t = 0$ and

$$\partial_x v_n (t, 0) + \beta_1 \varphi(t)\, v_n (t, 0) = 0.$$

Consequently, the corresponding integral is

$$\int_0^n \frac{2\beta_1 c(t)}{\varphi(t)})\partial_t v_n(t,0) v_n(t,0) dt = \frac{\beta_1 c(n)}{\varphi(n)} v_n^2(n,0) - \int_0^n \beta_1 (\frac{c(t)}{\varphi(t)})' v_n^2(t,0) dt,$$

which is negative thanks to the conditions (2.8) and the fact that $\beta_2 > 0$. On the part of the boundary where $x = 1$, we have $\nu_x = 1,\ \nu_t = 0$ and $v_n(t,1) = 0$. Consequently, the corresponding integral vanishes. Finally,

$$\begin{aligned} 2\int_{P_n} \tfrac{c(t)}{\varphi^2(t)} \partial_t v_n \partial_x^2 v_n dtdx &= -\int_0^1 \tfrac{c(n)}{\varphi^2(n)} (\partial_x v_n)^2 (n,x) dx + \tfrac{\beta_1 c(n)}{\varphi(n)} v_n^2(n,0) - \int_0^n \beta_1 (\tfrac{c(t)}{\varphi(t)})' v_n^2(t,0) dt \\ &\quad + \int_{P_n} (\tfrac{c(t)}{\varphi^2(t)})' (\partial_x v_n)^2 dtdx \end{aligned}$$

Note that the functions $\frac{c(t)}{\varphi^2(t)}$ and $(\frac{c(t)}{\varphi^2(t)})'$ are bounded. So, by using Lemma 2.3.1, we deduce

$$\begin{aligned} 2\int_{P_n} \tfrac{c(t)}{\varphi^2(t)} \partial_t v_n \partial_x^2 v_n dtdx &\leq \int_{P_n} (\tfrac{c(t)}{\varphi^2(t)})' (\partial_x v_n)^2 dtdx \\ &\leq K_2 \left\| \partial_x v_n \right\|_{L^2(P_n)}^2 \\ &\leq K_3 \left\| f_2 \right\|_{L^2(P)}^2 , \end{aligned}$$

where K_2 and K_3 are constants independent of n. Consequently, Choosing $\epsilon = \frac{1}{2}$ in the relationship (2.20), we obtain

$$\|\partial_t v_n\|^2_{L^2(P_n)} + \left\|\partial_x^2 v_n\right\|^2_{L^2(P_n)} \leq K \|f_2\|^2_{L^2(P)} .$$

This ends the proof of Proposition 2.3.1.

Remark 2.3.2. *We obtain the solution u of Problem (2.13) (respectively, v of Problem (2.14)) by letting n go to infinity in the previous proposition. The uniqueness can be proved as in Theorem 2.2.1.*

Finally, we have proved the following Theorem:

Theorem 2.3.1. *Problem (2.13) (respectively, Problem (2.14)) admits a (unique) solution $u \in \mathcal{H}^{1,2}(D_2)$ (respectively, $v \in \mathcal{H}^{1,2}(D_2)$).*

2.4 The case of a small in time bounded triangular domain

Let T be a small enough positive real number. We set

$$D_3 := \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T;\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

with $\varphi(0) = 0$ and consider the following problems:

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f_1 \ \text{ a.e. on } D_3, \\ u|_{\Gamma_1} = 0, \\ \partial_x u + \beta_2 u|_{\Gamma_2} = 0, \end{cases} \tag{2.21}$$

and

$$\begin{cases} \partial_t v - c(t)\partial_x^2 v = f_2 \ \text{ a.e. on } D_3, \\ v|_{\Gamma_2} = 0, \\ \partial_x v + \beta_1 v|_{\Gamma_1} = 0, \end{cases} \tag{2.22}$$

where $f_i \in L^2(D_3),\ i = 1,\ 2$. Set

$$Q_n = \left\{(t,x) \in D_3 : \frac{1}{n} < t < T\right\},\ n \in \mathbb{N}^* \text{ and } \frac{1}{n} < T.$$

For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, we set $f_i^{(n)} = f_i|_{Q_n} \in L^2(Q_n),\ i = 1, 2$ and denote by $u_n \in \mathcal{H}^{1,2}(Q_n)$ (respectively, $v_n \in \mathcal{H}^{1,2}(Q_n)$) the solution of the following problem:

$$\begin{cases} \partial_t u_n - c(t)\partial_x^2 u_n = f_1 \text{ a.e. on } Q_n, \\ u_n|_{t=\frac{1}{n}} = u_n|_{x=\varphi_1(t)} = 0, \\ \partial_x u_n + \beta_2 u_n|_{x=\varphi_2(t)} = 0. \end{cases} \tag{2.23}$$

(respectively,

$$\begin{cases} \partial_t v_n - c(t)\partial_x^2 v_n = f_2 \text{ a.e. on } Q_n, \\ v_n|_{t=\frac{1}{n}} = v_n|_{x=\varphi_2(t)} = 0, \\ \partial_x v_n + \beta_1 v_n|_{x=\varphi_1(t)} = 0). \end{cases} \tag{2.24}$$

Such a solution exists by Theorem 2.2.1.

Proposition 2.4.1. *There exists a constant $K > 0$ independent of n such that*

$$\|u_n\|^2_{\mathcal{H}^{1,2}(Q_n)} \leq K \left\|f_1^{(n)}\right\|^2_{L^2(Q_n)} \leq K \|f_1\|^2_{L^2(D_3)},$$

$$\|v_n\|^2_{\mathcal{H}^{1,2}(Q_n)} \leq K \left\|f_2^{(n)}\right\|^2_{L^2(Q_n)} \leq K \|f_2\|^2_{L^2(D_3)}.$$

Remark 2.4.1. *Let $\epsilon > 0$ be a real which we will choose small enough. The hypothesis (3.3) implies the existence of a real number $T > 0$ small enough such that*

$$|\varphi_i'(t)\,\varphi(t)| \leq \epsilon, \text{ for all } t \in (0,T),\ i = 1, 2. \tag{2.25}$$

In order to prove Proposition 2.4.1, we need some preliminary results.

Lemma 2.4.1. *There exists a constant K independent of n such that for all $t \in]0,T[$:*
1) $\|u_n\|_{L^2(Q_n)} \leq K \|\varphi\partial_x u_n\|_{L^2(Q_n)}, \quad \|v_n\|_{L^2(Q_n)} \leq K \|\varphi\partial_x v_n\|_{L^2(Q_n)};$
2) $\int_{\varphi_1(t)}^{\varphi_2(t)} u_n^2(t,x)dx \leq K\varphi^4 \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n)^2(t,x)dx, \quad \int_{\varphi_1(t)}^{\varphi_2(t)} v_n^2(t,x)dx \leq K\varphi^4 \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 v_n)^2(t,x)dx;$
3) $\int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2(t,x)dx \leq K\varphi^2 \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n)^2(t,x)dx, \quad \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x v_n)^2(t,x)dx \leq K\varphi^2 \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x^2 v_n)^2(t,x)dx;$
4) $\|\partial_x u_n\|_{L^2(Q_n)} \leq K \|f_1\|_{L^2(D_3)}, \quad \|\partial_x v_n\|_{L^2(Q_n)} \leq K \|f_2\|_{L^2(D_3)}.$

Proof. Inequalities (1) are consequences of the Poincaré inequality.

The following operators are isomorphisms (see, [15])

$$H_\gamma^2(0,1) \longrightarrow L^2(0,1), \; u \mapsto u'', \;\; H_\delta^2(0,1) \longrightarrow L^2(0,1), \; v \mapsto v'',$$

where,

$$H_\gamma^2(0,1) = \left\{ u \in H^2(0,1) : u(0) = 0, u'(1) + \beta_2 u(1) = 0 \right\}$$

and

$$H_\delta^2(0,1) = \left\{ v \in H^2(0,1) : v'(0) + \beta_1 v(0) = 0, v(1) = 0 \right\}.$$

So, there exists a constant $K > 0$ such that

$$\begin{array}{rclrcl} \|u\|_{L^2(0,1)} & \leq & \|u''\|_{L^2(0,1)}, & \|v\|_{L^2(0,1)} & \leq & \|v''\|_{L^2(0,1)}, \\ \|u'\|_{L^2(0,1)} & \leq & \|u''\|_{L^2(0,1)}, & \|v'\|_{L^2(0,1)} & \leq & \|v''\|_{L^2(0,1)}. \end{array}$$

The change of variables (for a fixed t)

$$[0,1] \to [\varphi_1(t), \varphi_2(t)]; \; x \longmapsto y = (1-x)\varphi_1(t) + x\varphi_2(t),$$

leads to the estimates (2) and (3).

To prove (4), it is sufficient to expand the inner products $\left\langle f_1^{(n)}, u_n \right\rangle$ and $\left\langle f_2^{(n)}, v_n \right\rangle$ and use the inequalities (1). Indeed, we deduce for all $\epsilon > 0$, (see the proof of uniqueness of solutions in Theorem 2.2.1)

$$\begin{array}{rcl} \int_{Q_n} c(t)(\partial_x u_n)^2 dt dx & \leq & \left| \left\langle f_1^{(n)}, u_n \right\rangle \right| \\ & \leq & \frac{1}{\epsilon} \left\| f_1^{(n)} \right\|_{L^2(Q_n)}^2 + \epsilon \|u_n\|_{L^2(Q_n)}^2 \\ & \leq & \frac{1}{\epsilon} \|f_1\|_{L^2(D_3)}^2 + \epsilon K \|\varphi u_n\|_{L^2(Q_n)}^2 \end{array}$$

and

$$\begin{array}{rcl} \int_{Q_n} c(t)(\partial_x v_n)^2 dt dx & \leq & \left| \left\langle f_2^{(n)}, v_n \right\rangle \right| \\ & \leq & \frac{1}{\epsilon} \left\| f_2^{(n)} \right\|_{L^2(Q_n)}^2 + \epsilon \|v_n\|_{L^2(Q_n)}^2 \\ & \leq & \frac{1}{\epsilon} \|f_2\|_{L^2(D_3)}^2 + \epsilon K \|\varphi v_n\|_{L^2(Q_n)}^2. \end{array}$$

However, φ is bounded and $c > \alpha > 0$. Choosing ϵ small enough yields the desired result. □

Proof of Proposition 2.4.1: Let us denote the inner product in $L^2(Q_n)$ by $\langle ., . \rangle$ and set $\mathcal{L} := \partial_t - c(t)\partial_x^2$, then we have

$$\left\| f_1^{(n)} \right\|_{L^2(Q_n)}^2 = \langle \mathcal{L}u_n, \mathcal{L}u_n \rangle = \|\partial_t u_n\|_{L^2(Q_n)}^2 + \|c(t)\partial_x^2 u_n\|_{L^2(Q_n)}^2 - 2\langle \partial_t u_n, c(t)\partial_x^2 u_n \rangle$$

and

$$\left\| f_2^{(n)} \right\|_{L^2(Q_n)}^2 = \langle \mathcal{L}v_n, \mathcal{L}v_n \rangle = \|\partial_t v_n\|_{L^2(Q_n)}^2 + \|c(t)\partial_x^2 v_n\|_{L^2(Q_n)}^2 - 2\langle \partial_t v_n, c(t)\partial_x^2 v_n \rangle.$$

Estimation of $-2\langle \partial_t u_n, c(t)\partial_x^2 u_n \rangle$: We have

$$\partial_t u_n \partial_x^2 u_n = \partial_x(\partial_t u_n \partial_x u_n) - \tfrac{1}{2}\partial_t \left(\partial_x u_n\right)^2.$$

Then,

$$\begin{aligned} -2\langle \partial_t u_n, c(t)\partial_x^2 u_n \rangle &= -2\int_{Q_n} c(t)\partial_t u_n \partial_x^2 u_n dtdx \\ &= -2\int_{Q_n} c(t)\partial_x \left(\partial_t u_n \partial_x u_n\right) dtdx + \int_{Q_n} c(t)\partial_t \left(\partial_x u_n\right)^2 dtdx \\ &= \int_{\partial Q_n} c(t) \left[\left(\partial_x u_n\right)^2 \nu_t - 2\partial_t u_n \partial_x u_n \nu_x\right] d\sigma - \int_{Q_n} c'(t)(\partial_x u_n)^2 dtdx \end{aligned}$$

where ν_t, ν_x are the components of the unit outward normal vector at the boundary of Q_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Q_n where $t = \frac{1}{n}$, we have $u_n = 0$ and consequently $\partial_x u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_{\varphi_1(T)}^{\varphi_2(T)} c(T) \left(\partial_x u_n\right)^2 dx$$

is nonnegative. On the parts of the boundary where $x = \varphi_i(t)$, $i = 1, 2$, we have

$$\nu_x = \frac{(-1)^i}{\sqrt{1 + (\varphi_i')^2(t)}}, \ \nu_t = \frac{(-1)^{i+1}\varphi_i'(t)}{\sqrt{1 + \left(\varphi_i'\right)^2(t)}}, \ u_n(t, \varphi_1(t)) = \partial_x u_n(t, \varphi_2(t)) + \beta_2 u_n(t, \varphi_2(t)) = 0.$$

Consequently, the corresponding integral is

$$\begin{aligned} &-\int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t) \left[\partial_x u_n(t, \varphi_1(t))\right]^2 dt - 2\int_{\frac{1}{n}}^{T} c(t)\partial_t u_n(t, \varphi_2(t)) . \partial_x u_n(t, \varphi_2(t)) \, dt \\ &-\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t) \left[\partial_x u_n(t, \varphi_2(t))\right]^2 dt. \end{aligned}$$

By putting $h(t) := u_n(t, \varphi_2(t)), \ t \in [\frac{1}{n}, T]$, we obtain

$$\partial_t u_n(t, \varphi_2(t))\partial_x u_n(t, \varphi_2(t)) = h'(t)\partial_x u_n(t, \varphi_2(t)) - \varphi_2'(t) \left(\partial_x u_n(t, \varphi_2(t))\right)^2.$$

So, by using the boundary conditions, we get

$$
\begin{aligned}
&-2\int_{\frac{1}{n}}^{T} c(t)\partial_t u_n(t,\varphi_2(t))\partial_x u_n(t,\varphi_2(t))\,dt\\
&\quad= -2\int_{\frac{1}{n}}^{T} c(t)h'(t)\partial_x u_n(t,\varphi_2(t))\,dt + 2\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t)\left(\partial_x u_n(t,\varphi_2(t))\right)^2\,dt\\
&\quad= 2\beta_2\int_{\frac{1}{n}}^{T} c(t)h'(t)h(t)\,dt + 2\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t)\left(\partial_x u_n(t,\varphi_2(t))\right)^2\,dt\\
&\quad= \beta_2\int_{\frac{1}{n}}^{T} c(t)(h(t)^2)'\,dt + 2\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t)\left(\partial_x u_n(t,\varphi_2(t))\right)^2\,dt\\
&\quad= \beta_2 c(T)(h(T))^2 - \beta_2\int_{\frac{1}{n}}^{T} c'(t)u_n^2(t,\varphi_2(t)\,dt + 2\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t)\left(\partial_x u_n(t,\varphi_2(t))\right)^2\,dt.
\end{aligned}
$$

Observe that, thanks to the condition (2.8) and the fact that $\beta_2 > 0,\ c(t) > 0$, we have

$$\beta_2 c(T)(h(T))^2 - \beta_2\int_{\frac{1}{n}}^{T} c'(t)u_n^2(t,\varphi_2(t)\,dt \;\geq\; 0.$$

So, by setting

$$
\begin{aligned}
I_{n,1} &= -\int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t)\left[\partial_x u_n(t,\varphi_1(t))\right]^2 dt,\\
I_{n,2} &= \int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t)\left[\partial_x u_n(t,\varphi_2(t))\right]^2 dt,
\end{aligned}
$$

we have

$$-2\langle \partial_t u_n, c(t)\partial_x^2 u_n\rangle \geq -\left|I_{n,1}\right| - \left|I_{n,2}\right|. \tag{2.26}$$

Estimation of $-2\langle \partial_t v_n, c(t)\partial_x^2 v_n\rangle$: We have

$$\partial_t v_n\partial_x^2 v_n = \partial_x\left(\partial_t v_n\partial_x v_n\right) - \tfrac{1}{2}\partial_t\left(\partial_x v_n\right)^2.$$

Then,

$$
\begin{aligned}
-2\langle \partial_t u_n, c(t)\partial_x^2 u_n\rangle &= -2\int_{Q_n} c(t)\partial_t v_n\partial_x^2 v_n dtdx\\
&= -2\int_{Q_n} c(t)\partial_x\left(\partial_t v_n\partial_x v_n\right)dtdx + \int_{Q_n} c(t)\partial_t\left(\partial_x v_n\right)^2 dtdx\\
&= \int_{\partial Q_n} c(t)\left[\left(\partial_x v_n\right)^2\nu_t - 2\partial_t v_n\partial_x v_n\nu_x\right]d\sigma - \int_{Q_n} c'(t)(\partial_x v_n)^2 dtdx
\end{aligned}
$$

where ν_t, ν_x are the components of the unit outward normal vector at the boundary of Q_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Q_n where $t = \frac{1}{n}$, we have $v_n = 0$ and consequently $\partial_x v_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_{\varphi_1(T)}^{\varphi_2(T)} c(T) \left(\partial_x v_n\right)^2 dx$$

is nonnegative. On the parts of the boundary where $x = \varphi_i\left(t\right), \, i = 1, 2$, we have

$$\nu_x = \frac{\left(-1\right)^i}{\sqrt{1 + \left(\varphi_i'\right)^2 \left(t\right)}}, \; \nu_t = \frac{\left(-1\right)^{i+1} \varphi_i'\left(t\right)}{\sqrt{1 + \left(\varphi_i'\right)^2 \left(t\right)}}, \; v_n\left(t, \varphi_2\left(t\right)\right) = \partial_x v_n\left(t, \varphi_1\left(t\right)\right) + \beta_1 v_n\left(t, \varphi_1\left(t\right)\right) = 0.$$

Consequently, the corresponding integral is

$$\int_{\frac{1}{n}}^{T} c(t)\varphi_2'\left(t\right) \left[\partial_x v_n\left(t, \varphi_2\left(t\right)\right)\right]^2 dt + 2\int_{\frac{1}{n}}^{T} c(t)\partial_t v_n\left(t, \varphi_1\left(t\right)\right).\partial_x v_n\left(t, \varphi_1\left(t\right)\right) dt$$
$$\int_{\frac{1}{n}}^{T} c(t)\varphi_1'\left(t\right) \left[\partial_x v_n\left(t, \varphi_1\left(t\right)\right)\right]^2 dt.$$

By putting $k(t) := v_n(t, \varphi_1(t)), \; t \in [\frac{1}{n}, T]$, we obtain

$$\partial_t v_n(t, \varphi_1(t))\partial_x v_n(t, \varphi_1(t)) = k'(t)\partial_x v_n(t, \varphi_1(t)) - \varphi_1'(t)\left(\partial_x v_n(t, \varphi_1(t))\right)^2.$$

So, by using the boundary conditions, we get

$$\begin{aligned}
&2\textstyle\int_{\frac{1}{n}}^{T} c(t)\partial_t v_n(t, \varphi_1(t))\partial_x v_n(t, \varphi_1(t))\, dt \\
&\quad = 2\int_{\frac{1}{n}}^{T} c(t)k'(t)\partial_x v_n(t, \varphi_1(t))\, dt - 2\int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t)\left(\partial_x v_n(t, \varphi_1(t))\right)^2 dt \\
&\quad = -2\beta_1\int_{\frac{1}{n}}^{T} c(t)k'(t)k(t)\, dt - 2\int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t)\left(\partial_x v_n(t, \varphi_1(t))\right)^2 dt \\
&\quad = -\beta_1\int_{\frac{1}{n}}^{T} c(t)(k(t)^2)'\, dt - 2\int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t)\left(\partial_x v_n(t, \varphi_1(t))\right)^2 dt \\
&\quad = -\beta_1 c(T)(k(T))^2 + \beta_1\int_{\frac{1}{n}}^{T} c'(t)v_n^2(t, \varphi_1(t)\, dt - 2\int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t)\left(\partial_x v_n(t, \varphi_1(t))\right)^2 dt.
\end{aligned}$$

Observe that, thanks to the condition (2.8) and the fact that $\beta_1 < 0,\ c(t) > 0$, we have

$$-\beta_1 c(T)(k(T))^2 + \beta_1 \int_{\frac{1}{n}}^{T} c'(t)v_n^2(t,\varphi_1(t)\,dt \quad \geq \quad 0.$$

So, by setting

$$\begin{aligned} J_{n,1} &= \int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t)\left[\partial_x v_n(t,\varphi_1(t))\right]^2 dt, \\ J_{n,2} &= -\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t)\left[\partial_x v_n(t,\varphi_2(t))\right]^2 dt, \end{aligned}$$

we have

$$-2\langle \partial_t v_n, c(t)\partial_x^2 v_n\rangle \geq -\left|J_{n,1}\right| - \left|J_{n,2}\right|. \tag{2.27}$$

Estimation of $I_{n,k},\ J_{n,k},\ k=1,\ 2:$

Lemma 2.4.2. *There exists a constant $K > 0$ independent of n such that*

$$\begin{aligned} \max(\left|I_{n,1}\right|,\left|I_{n,2}\right|) &\leq K\epsilon \left\|\partial_x^2 u_n\right\|^2_{L^2(Q_n)}, \\ \max(\left|J_{n,1}\right|,\left|J_{n,2}\right|) &\leq K\epsilon \left\|\partial_x^2 v_n\right\|^2_{L^2(Q_n)}. \end{aligned}$$

Proof. We convert the boundary integral $J_{n,1}$ into a surface integral by setting

$$\begin{aligned} \left[\partial_x v_n(t,\varphi_1(t))\right]^2 &= -\tfrac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}\left[\partial_x v_n(t,x)\right]^2\Big|_{x=\varphi_1(t)}^{x=\varphi_2(t)} \\ &= -\textstyle\int_{\varphi_1(t)}^{\varphi_2(t)} \frac{\partial}{\partial x}\left\{\frac{\varphi_2(t)-x}{\varphi(t)}\left[\partial_x v_n(t,x)\right]^2\right\} dx \\ &= -2\textstyle\int_{\varphi_1(t)}^{\varphi_2(t)} \frac{\varphi_2(t)-x}{\varphi(t)}\partial_x v_n(t,x)\,\partial_x^2 v_n(t,x)\,dx + \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{1}{\varphi(t)}\left[\partial_x v_n(t,x)\right]^2 dx. \end{aligned}$$

Then, we have

$$\begin{aligned} J_{n,1} &= \textstyle\int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t)\left[\partial_x v_n(t,\varphi_1(t))\right]^2 dt \\ &= \textstyle\int_{Q_n} \frac{c(t)\varphi_1'(t)}{\varphi(t)}\left(\partial_x v_n\right)^2 dtdx - 2\int_{Q_n} \frac{\varphi_2(t)-x}{\varphi(t)} c(t)\varphi_1'(t)\left(\partial_x v_n\right)\left(\partial_x^2 v_n\right) dtdx. \end{aligned}$$

Thanks to Lemma 2.4.1, we can write

$$\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x v_n(t,x)\right]^2 dx \leq C\left[\varphi(t)\right]^2 \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x^2 v_n(t,x)\right]^2 dx.$$

Therefore,

$$\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x v_n(t,x)\right]^2 \frac{\left|\varphi_1'\right|}{\varphi} dx \leq C\left|\varphi_1'\right|\left[\varphi\right] \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x^2 v_n(t,x)\right]^2 dx,$$

consequently,

$$|J_{n,1}| \leq C\int_{Q_n} c(t)\,|\varphi_1'|\,[\varphi]\,(\partial_x^2 v_n)^2\,dtdx + 2\int_{Q_n} c(t)\,|\varphi_1'|\,|\partial_x v_n|\,|\partial_x^2 v_n|\,dtdx,$$

since $\left|\frac{\varphi_2(t)-x}{\varphi(t)}\right| \leq 1$. So, for all $\epsilon > 0$, we have

$$|J_{n,1}| \leq C\int_{Q_n} |c(t)\varphi_1'|\,[\varphi]\,(\partial_x^2 v_n)^2\,dtdx + \epsilon\int_{Q_n} c(t)\,(\partial_x^2 v_n)^2\,dtdx + \tfrac{1}{\epsilon}\int_{Q_n} c(t)\,(\varphi_1')^2\,(\partial_x v_n)^2\,dtdx.$$

Lemma 2.4.1 yields

$$\tfrac{1}{\epsilon}\int_{Q_n} c(t)\,(\varphi_1')^2\,(\partial_x v_n)^2\,dtdx \leq C\tfrac{1}{\epsilon}\int_{Q_n} c(t)\,(\varphi_1')^2\,[\varphi]^2\,(\partial_x^2 v_n)^2\,dtdx.$$

Thus, there exists a constant $M > 0$ independent of n such that

$$\begin{aligned}|J_{n,1}| &\leq C\int_{Q_n} c(t)\left[|\varphi_1'|\,|\varphi| + \tfrac{1}{\epsilon}(\varphi_1')^2\,|\varphi|^2\right](\partial_x^2 v_n)^2\,dtdx + \epsilon\int_{Q_n} c(t)\,(\partial_x^2 v_n)^2\,dtdx \\ &\leq M\epsilon\int_{Q_n} (\partial_x^2 v_n)^2\,dtdx,\end{aligned}$$

because $\left|\varphi_1'\varphi\right| \leq \epsilon$. The inequalities

$$|I_{n,1}| \leq K\epsilon\left\|\partial_x^2 u_n\right\|_{L^2(Q_n)}^2,\ |I_{n,2}| \leq K\epsilon\left\|\partial_x^2 u_n\right\|_{L^2(Q_n)}^2,\ |J_{n,2}| \leq K\epsilon\left\|\partial_x^2 v_n\right\|_{L^2(Q_n)}^2,$$

can be proved by a similar argument. □

Now, we can complete the proof of Proposition 2.4.1. Summing up the estimates (2.26), (2.27) and those of Lemma 2.4.2, we then obtain

$$\begin{aligned}\left\|f_1^{(n)}\right\|_{L^2(Q_n)}^2 &\geq \|\partial_t u_n\|_{L^2(Q_n)}^2 + \|c(t)\partial_x^2 u_n\|_{L^2(Q_n)}^2 - K_1\epsilon\,\|\partial_x^2 u_n\|_{L^2(Q_n)}^2 \\ &\geq \|\partial_t u_n\|_{L^2(Q_n)}^2 + (\alpha^2 - K_1\epsilon)\,\|\partial_x^2 u_n\|_{L^2(Q_n)}^2\end{aligned}$$

and

$$\begin{aligned}\left\|f_2^{(n)}\right\|_{L^2(Q_n)}^2 &\geq \|\partial_t v_n\|_{L^2(Q_n)}^2 + \|c(t)\partial_x^2 v_n\|_{L^2(Q_n)}^2 - K_2\epsilon\,\|\partial_x^2 v_n\|_{L^2(Q_n)}^2 \\ &\geq \|\partial_t v_n\|_{L^2(Q_n)}^2 + (\alpha^2 - K_2\epsilon)\,\|\partial_x^2 v_n\|_{L^2(Q_n)}^2,\end{aligned}$$

where $K_i,\ i = 1,2$ are positive numbers. Then, it is sufficient to choose ϵ such that

$$\min\left(\alpha^2 - K_1\epsilon, \alpha^2 - K_2\epsilon\right) > 0,$$

to get a constant $K_0 > 0$ independent of n such that

$$\left\|f_1^{(n)}\right\|_{L^2(Q_n)}^2 \geq K_0(\|\partial_t u_n\|_{L^2(Q_n)}^2 + \left\|\partial_x^2 u_n\right\|_{L^2(Q_n)}^2)$$

and

$$\left\| f_2^{(n)} \right\|^2_{L^2(Q_n)} \geq K_0(\|\partial_t v_n\|^2_{L^2(Q_n)} + \left\|\partial_x^2 v_n\right\|^2_{L^2(Q_n)}).$$

But

$$\begin{aligned} \left\| f_1^{(n)} \right\|_{L^2(Q_n)} &\leq \|f_1\|_{L^2(D_3)}, \\ \left\| f_2^{(n)} \right\|_{L^2(Q_n)} &\leq \|f_2\|_{L^2(D_3)}, \end{aligned}$$

then, there exists a constant $K > 0$, independent of n satisfying

$$\|\partial_t u_n\|^2_{L^2(Q_n)} + \left\|\partial_x^2 u_n\right\|^2_{L^2(Q_n)} \leq K \|f_1\|^2_{L^2(D_3)}$$

and

$$\|\partial_t v_n\|^2_{L^2(Q_n)} + \left\|\partial_x^2 v_n\right\|^2_{L^2(Q_n)} \leq K \|f_2\|^2_{L^2(D_3)}.$$

Consequently, making use of Lemma 2.4.1 and the previous estimates, then, there exists a constant $K > 0$, independent of n satisfying

$$\|u_n\|^2_{\mathcal{H}^{1,2}(Q_n)} \leq C \|f_1\|^2_{L^2(D_3)}$$

and

$$\|v_n\|^2_{\mathcal{H}^{1,2}(Q_n)} \leq C \|f_2\|^2_{L^2(D_3)}.$$

This ends the proof of Proposition 2.4.1. Finally, we have proved the following Theorem:

Theorem 2.4.1. *Problem (2.21) (respectively, (2.22)) admits a (unique) solution $u \in \mathcal{H}^{1,2}(D_3)$ (respectively, $v \in \mathcal{H}^{1,2}(D_3)$).*

Proof. We obtain the solution u of Problem (2.21) (respectively, v of Problem (2.22)) by letting n go to infinity in the previous proposition. The uniqueness can be proved as in Theorem 2.2.1. □

2.5 Back to Problems (2.1)-(2.3) and (2.2)-(2.4) and proof of Theorem 2.1.1

The proof of Theorem 2.1.1 can be obtained by subdividing the domain

$$D := \left\{(t,x) \in \mathbb{R}^2 : t > 0,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

into three open subdomains $\Omega_1,\ \Omega_2$ and Ω_3. So, we set $D = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Gamma_{T_1} \cup \Gamma_{T_2}$ where

$$\Omega_1 = \{(t,x) \in D : 0 < t < T_1\},\ \Omega_2 = \{(t,x) \in D : T_1 < t < T_2\},\ \Omega_3 = \{(t,x) \in D : t > T_2\},$$

$$\Gamma_{T_1} = \left\{(T_1,x) \in \mathbb{R}^2 : \varphi_1(T_1) < x < \varphi_2(T_1)\right\} \text{ and } \Gamma_{T_2} = \left\{(T_2,x) \in \mathbb{R}^2 : \varphi_1(T_2) < x < \varphi_2(T_2)\right\}$$

with T_1 is a small enough positive number and T_2 is an arbitrary positive number such that $T_2 > T_1$. In the sequel, $f_1,\ f_2$ stands for an arbitrary fixed elements of $L^2(D)$ and $(f_1^{(i)}, f_2^{(i)}) = (f_1|_{\Omega_i}, f_2|_{\Omega_i}),\ i = 1,2,3.$

Theorem 3.4.1 applied to the triangular domain Ω_1, shows that there exists a unique solution $w_1 \in \mathcal{H}^{1,2}(\Omega_1)$ (respectively, $w_2 \in \mathcal{H}^{1,2}(\Omega_1)$) of the problem

$$\begin{cases} \partial_t w_1 - c(t)\partial_x^2 w_1 = f_1^{(1)} \text{ a.e. on } \Omega_1, \\ w_1|_{\Gamma_{1,1}} = 0, \\ \partial_x w_1 + \beta_2 w_1|_{\Gamma_{2,1}} = 0, \end{cases} \tag{2.28}$$

(respectively,

$$\begin{cases} \partial_t w_2 - c(t)\partial_x^2 w_2 = f_2^{(1)} \text{ a.e. on } \Omega_1, \\ w_2|_{\Gamma_{2,1}} = 0, \\ \partial_x w_2 + \beta_1 w_2|_{\Gamma_{1,1}} = 0), \end{cases} \tag{2.29}$$

where $f_i^{(1)} \in L^2(\Omega_1),\ i = 1,2$ and $\Gamma_{i,1}$ are the parts of the boundary of Ω_1 where $x = \varphi_i(t),\ i = 1,2.$

Lemma 2.5.1. *If $w \in \mathcal{H}^{1,2}(]0,T[\times]0,1[)$, then $w|_{t=0} \in H^1(\gamma_0)$, $w|_{x=0} \in H^{\frac{3}{4}}(\gamma_1)$ and $w|_{x=1} \in H^{\frac{3}{4}}(\gamma_2)$, where $\gamma_0 = \{0\}\times]0,1[$, $\gamma_1 =]0,T[\times\{0\}$ and $\gamma_2 =]0,T[\times\{1\}$.*

It is a particular case of Theorem 2.1 ([21], Vol.2). The transformation

$$(t,x) \longmapsto (t',x') = (t, \varphi(t)\,x + \varphi_1(t))$$

leads to the following lemma:

Lemma 2.5.2. *If $w \in \mathcal{H}^{1,2}(\Omega_2)$, then $w|_{\Gamma_{T_1}} \in H^1(\Gamma_{T_1})$, $w|_{x=\varphi_1(t)} \in H^{\frac{3}{4}}(\Gamma_{1,2})$ and $w|_{x=\varphi_2(t)} \in H^{\frac{3}{4}}(\Gamma_{2,2})$, where $\Gamma_{i,2}$ are the parts of the boundary of Ω_2 where $x = \varphi_i(t)$, $i = 1,2$.*

Hereafter, we denote the trace $w_1|_{\Gamma_{T_1}}$ (respectively, $w_2|_{\Gamma_{T_1}}$) by ψ_1 (respectively, ψ_2) which is in the Sobolev space $H^1(\Gamma_{T_1})$ because $w_1 \in \mathcal{H}^{1,2}(\Omega_1)$ (respectively, $w_2 \in \mathcal{H}^{1,2}(\Omega_1)$) (see Lemma 2.5.2). Now, consider the following problem in Ω_2 :

$$\begin{cases} \partial_t w_3 - c(t)\partial_x^2 w_3 = f_1^{(2)} \text{ a.e. on } \Omega_2, \\ w_3|_{\Gamma_{T_1}} = \psi_1, \\ w_3|_{\Gamma_{1,2}} = 0, \\ \partial_x w_3 + \beta_2 w_3|_{\Gamma_{2,2}} = 0, \end{cases} \tag{2.30}$$

(respectively,

$$\begin{cases} \partial_t w_4 - c(t)\partial_x^2 w_4 = f_2^{(2)} \text{ a.e. on } \Omega_2, \\ w_4|_{\Gamma_{T_1}} = \psi_2, \\ w_4|_{\Gamma_{2,2}} = 0, \\ \partial_x w_4 + \beta_1 w_4|_{\Gamma_{1,2}} = 0, \end{cases} \tag{2.31}$$

where $f_i^{(2)} \in L^2(\Omega_2)$, $i = 1, 2$ and $\Gamma_{i,2}$ are the parts of the boundary of Ω_2 where $x = \varphi_i(t)$, $i = 1, 2$. We use the following result, which is a consequence of Theorem 4.3 ([21], Vol.2), to solve Problem (2.30).

Proposition 2.5.1. *Let Q be the rectangle $]0, T[\times]0, 1[$, $f_1, f_2 \in L^2(Q)$ and $\psi_1, \psi_2 \in H^1(\gamma_0)$. Then, the following problem admits a (unique) solution $u \in \mathcal{H}^{1,2}(Q)$ (respectively, $v \in \mathcal{H}^{1,2}(Q)$):*

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f_1 \in L^2(Q), \\ u|_{\gamma_0} = \psi_1, \\ u|_{\gamma_1} = 0, \\ \partial_x u + \beta_2 u|_{\gamma_2} = 0, \end{cases}$$

(respectively,

$$\begin{cases} \partial_t v - c(t)\partial_x^2 v = f_2 \in L^2(Q), \\ v|_{\gamma_0} = \psi_2, \\ v|_{\gamma_2} = 0, \\ \partial_x v + \beta_1 v|_{\gamma_1} = 0), \end{cases}$$

where $\gamma_0 = \{0\} \times]0, 1[$, $\gamma_1 =]0, T[\times\{0\}$ and $\gamma_2 =]0, T[\times\{1\}$.

Thanks to the transformation

$$(t,x) \longmapsto (t,y) = (t, \varphi(t)\, x + \varphi_1(t)),$$

we deduce the following result:

Proposition 2.5.2. *Problem (2.30) (respectively, (2.31)) admits a (unique) solution $w_3 \in \mathcal{H}^{1,2}(\Omega_2)$ (respectively, $w_4 \in \mathcal{H}^{1,2}(\Omega_2)$).*

Hereafter, we denote the trace $w_3|_{\Gamma_{T_2}}$ by Φ_1 (respectively, $w_4|_{\Gamma_{T_2}}$ by Φ_2) which is in the Sobolev space $H^1(\Gamma_{T_2})$ because $w_3 \in \mathcal{H}^{1,2}(\Omega_2)$ (respectively, $w_4 \in \mathcal{H}^{1,2}(\Omega_2)$) (see Lemma 2.5.2). Now, consider the following problem in Ω_3 :

$$\begin{cases} \partial_t w_5 - c(t)\partial_x^2 w_5 = f_1^{(3)} \text{ a.e. on } \Omega_3, \\ w_5|_{\Gamma_{T_2}} = \Phi_1, \\ w_5|_{\Gamma_{1,3}} = 0, \\ \partial_x w_5 + \beta_2 w_5|_{\Gamma_{2,3}} = 0, \end{cases} \tag{2.32}$$

(respectively,

$$\begin{cases} \partial_t w_6 - c(t)\partial_x^2 w_6 = f_2^{(3)} \text{ a.e. on } \Omega_3, \\ w_6|_{\Gamma_{T_2}} = \Phi_2, \\ w_6|_{\Gamma_{2,3}} = 0, \\ \partial_x w_6 + \beta_1 w_6|_{\Gamma_{1,3}} = 0,) \end{cases} \tag{2.33}$$

where $f_i^{(3)} \in L^2(\Omega_3)$, $i = 1, 2$ and $\Gamma_{i,3}$ are the parts of the boundary of Ω_3 where $x = \varphi_i(t)$, $i = 1, 2$. By similar arguments like those used previously, we deduce the following result:

Proposition 2.5.3. *Problem (2.32) (respectively, (2.33)) admits a (unique) solution $w_5 \in \mathcal{H}^{1,2}(\Omega_3)$ (respectively, $w_6 \in \mathcal{H}^{1,2}(\Omega_3)$).*

Finally, the function u (respectively, v) defined by

$$u := \begin{cases} w_1 \text{ in } \Omega_1, \\ w_3 \text{ in } \Omega_2, \\ w_5 \text{ in } \Omega_3, \end{cases}$$

(respectively,

$$v := \begin{cases} w_2 \text{ in } \Omega_1, \\ w_4 \text{ in } \Omega_2, \\ w_6 \text{ in } \Omega_3, \end{cases}$$

is the (unique) solution of Problem (2.1)-(2.3) (respectively, (2.2)-(2.4)). This ends the proof of Theorem 2.1.1.

CHAPTER 3

RESOLUTION OF A PARABOLIC EQUATION IN UNBOUNDED CONICAL DOMAINS OF $\mathbb{R}^3$

3.1 Introduction and statement of the main result

Let D be an open set of $\mathbb{R}^3$ defined by

$$D = \left\{ (t, x, y) \in \mathbb{R}^3 : t > 0;\ 0 \leq \sqrt{x^2 + y^2} < \varphi\left(t\right) \right\}$$

where $\varphi \in C([0, +\infty[) \cap C^1(0, +\infty)$,

$$\varphi\left(t\right) > 0 \quad \forall t > 0, \text{ and } \varphi\left(0\right) = 0.$$

Let us introduce the following functional space

$$\mathcal{H}^{1,2}\left(D\right) = \left\{ u \in H^1(D) : \partial_x^2 u \in L^2\left(D\right), \partial_y^2 u \in L^2\left(D\right), \partial_{xy}^2 u \in L^2\left(D\right) \right\}$$

where $H^1(D)$ stands for the Sobolev space defined by

$$H^1\left(D\right) = \left\{ u \in L^2(D) : \partial_t u \in L^2\left(D\right), \partial_x u \in L^2\left(D\right), \partial_y u \in L^2\left(D\right) \right\}$$

with $L^2(D)$ stands for the usual Lebesgue space of square-integrable functions on D. The space $\mathcal{H}^{1,2}(D)$ is equipped with the natural norm, that is

$$\|u\|_{\mathcal{H}^{1,2}(D)} = \left(\|u\|^2_{H^1(D)} + \left\|\partial_x^2 u\right\|^2_{L^2(D)} + \left\|\partial_{xy}^2 u\right\|^2_{L^2(D)} + \left\|\partial_y^2 u\right\|^2_{L^2(D)} \right)^{1/2}.$$

We consider the problem: to find a function $u \in \mathcal{H}^{1,2}(D)$ that satisfies the equation

$$\partial_t u - c(t)(\partial_x^2 u + \partial_y^2 u) = f \text{ a.e. on } D \tag{3.1}$$

and the boundary condition

$$u|_{\partial D} = 0, \tag{3.2}$$

where $f \in L^2(D)$ and the coefficient c is a continuous real-valued function defined on $[0, +\infty[$, differentiable on $]0, +\infty[$ and such that

$$0 < \alpha \leq c(t) \leq \beta$$

for every $t \in [0, +\infty[$, where α and β are positive constants.

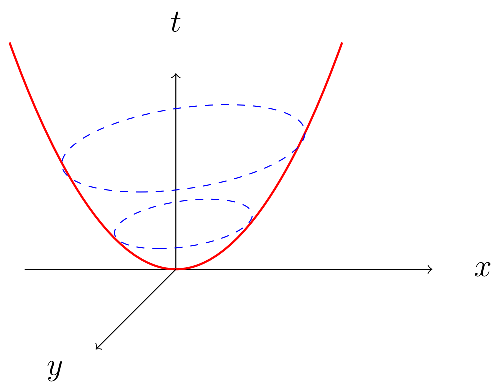

Fig.1: The unbounded conical domain D.

Observe that the main difficulties related to this kind of problems are due to the facts that $\varphi(0) = 0$ and the unboundedness of D with respect to the time variable. It is well known that there are two main approaches for the study of our problem. We can look for boundary conditions assuring the existence of the solution in the natural space, or we can work directly in the 'bad' domain which generates some singularities in the solution (see, for example [25] and

[12]). It is the first approach that we follow in this work. So, we impose sufficient conditions on the function φ in the neighborhoods of 0 and $+\infty$. Our main result is the following :

Theorem 3.1.1. *Let us assume that*

$$|\varphi'(t)|\,\varphi(t) \quad \rightarrow \quad 0 \qquad \textit{as } t \rightarrow 0^{+}, \tag{3.3}$$

$$\varphi \textit{ and } \varphi' \textit{ are bounded in a neighborhood of} +\infty, \tag{3.4}$$

and one of the following conditions is satisfied

(a) *φ is increasing in a neighborhood of $+\infty$,*

(b) $\exists M > 0 : |\varphi'|\,\varphi \leq Mc(t)$.

Then Problem (3.1),(3.2) admits a unique solution $u \in \mathcal{H}^{1,2}(D)$.

3.2 Problem (3.1),(3.2) in a bounded domain which can be transformed into a cylinder

Let T be an arbitrary positive number. Denote by

$$D_1 := \left\{(t,x,y) \in \mathbb{R}^3 : 0 < t < T;\ 0 < \sqrt{x^2+y^2} < \varphi(t)\right\}$$

with $\varphi(t) > 0$ for all $t \in [0,T]$ and consider the following problem: to find a function $u \in \mathcal{H}^{1,2}(D_1)$ that satisfies the equation

$$\partial_t u - c(t)(\partial_x^2 u + \partial_y^2 u) = f \ \text{ a.e. on } D_1 \tag{3.5}$$

and the boundary condition

$$u|_{\partial D_1 \setminus \Gamma_T} = 0, \tag{3.6}$$

where $f \in L^2(D_1)$ and Γ_T is the part of ∂D_1 where $t = T$, defined by

$$\Gamma_T := \left\{(T,x,y) \in \mathbb{R}^3 : 0 < \sqrt{x^2+y^2} < \varphi(T)\right\}.$$

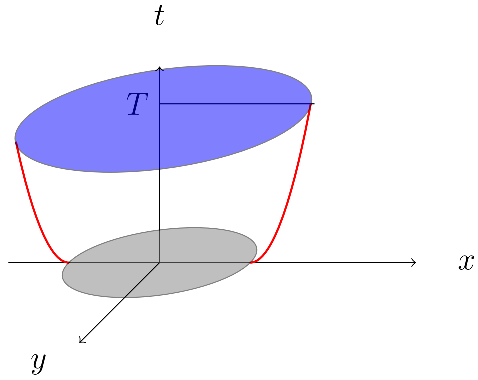

Fig.2: The bounded domain D_1.

Let us denote the inner product in $L^2(D_1)$ by $\langle .,. \rangle$. Then, the uniqueness of the solutions may be obtained by developing the inner product $\left\langle \partial_t u - c(t)(\partial_x^2 u + \partial_y^2 u), u \right\rangle$. Indeed, Let us consider $u \in \mathcal{H}^{1,2}(D_1)$ a solution of Problem (3.5),(3.6) with a null right-hand side term. So,

$$\partial_t u - c(t)(\partial_x^2 u + \partial_y^2 u) = 0 \text{ in } D_1.$$

In addition u fulfils the boundary condition (3.6). Using Green's formula, we have

$$\begin{aligned} &\textstyle\int_{D_1} \left[\partial_t u - c(t)(\partial_x^2 u + \partial_y^2 u)\right] u \; dt \; dx \; dy \\ &= \textstyle\int_{\partial D_1} \left[\frac{1}{2} |u|^2 \nu_t - c(t)(u.\partial_x u \nu_x + u.\partial_y u \nu_y)\right] d\sigma \\ &+ \textstyle\int_{D_1} c(t)(|\partial_x u|^2 + |\partial_x u|^2) dt \; dx \; dy, \end{aligned}$$

where ν_t, ν_x, ν_y are the components of the unit outward normal vector at ∂D_1. Taking into account the boundary conditions, all the boundary integrals vanish except $\int_{\partial D_1} |u|^2 \nu_t d\sigma$. We have

$$\int_{\partial D_1} |u|^2 \nu_t d\sigma = \int_{\Gamma_T} |u|^2 \, dxdy.$$

Then

$$\begin{aligned} &\textstyle\int_{D_1} \left[\partial_t u - c(t)(\partial_x^2 u + \partial_y^2 u)\right] u \; dt \; dx \; dy \\ &= \textstyle\int_{\Gamma_T} \frac{1}{2} |u|^2 \, dx \; dy + \int_{D_1} c(t)(|\partial_x u|^2 + |\partial_y u|^2) dt \; dx \; dy. \end{aligned}$$

Consequently

$$\int_{D_1} \left[\partial_t u - c(t)(\partial_x^2 u + \partial_y^2 u)\right] u \; dt \; dx \; dy = 0$$

yields

$$\int_{D_1} c(t)(|\partial_x u|^2 + |\partial_y u|^2)dt\ dx\ dy = 0,$$

because

$$\int_{\Gamma_T} \frac{1}{2}\,|u|^2\,dx\ dy \geq 0.$$

This implies that $|\partial_x u|^2+|\partial_y u|^2 = 0$, since $c(t) > 0$ for all $t \in [0,T]$. Consequently, $\partial_x^2 u+\partial_y^2 u = 0$. Then, the hypothesis $\partial_t u - c(t)(\partial_x^2 u + \partial_y^2 u) = 0$ gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions imply that $u = 0$ in D_1. This proves the uniqueness of the solution of Problem (3.5),(3.6).

Now, let us look at the existence of solutions for Problem (3.5),(3.6). The change of variables (t, x, y) to $\left(t, \frac{x}{\varphi(t)}, \frac{y}{\varphi(t)}\right)$ transforms D_1 into the cylinder $Q =]0, T[\, \times D\,(0, 1)$, where $D\,(0, 1)$ is the unit disk of $\mathbb{R}^2$ and Problem (3.5),(3.6) becomes the following: to find a function $u \in \mathcal{H}^{1,2}(Q)$ that satisfies the equation

$$\partial_t u + (a\,(t, x)\,\partial_x u + a\,(t, y)\,\partial_y u) - b\,(t)\,(\partial_x^2 u + \partial_y^2 u) = f \ \text{ a.e. on } Q \tag{3.7}$$

and the boundary condition

$$u|_{\partial Q\setminus(\{T\}\times D(0,1))} = 0, \tag{3.8}$$

where $f \in L^2(Q)$ and

$$\begin{aligned} &a\,(t, x) = -\tfrac{\varphi'(t)}{\varphi(t)}x,\ a\,(t, y) = -\tfrac{\varphi'(t)}{\varphi(t)}y, \\ &b\,(t) = \tfrac{c(t)}{\varphi^2(t)}. \end{aligned}$$

The coefficients $a(t, x)$ and $a(t, y)$ are bounded. So, the operator

$$[a(t, x)\partial_x + a(t, y)\partial_y] : \mathcal{H}^{1,2}\,(Q) \longrightarrow L^2\,(Q)$$

is compact. Hence, it is sufficient to study the following problem: to find a function $u \in \mathcal{H}^{1,2}(Q)$ that satisfies the equation

$$\partial_t u - b\,(t)\,(\partial_x^2 u + \partial_y^2 u) = f \ \text{ a.e. on } Q \tag{3.9}$$

and the boundary condition

$$u|_{\partial Q\setminus(\{T\}\times D(0,1))} = 0. \tag{3.10}$$

It is clear that Problem (3.9),(3.10) admits a (unique) solution $u \in \mathcal{H}^{1,2}(Q)$ because the coefficient b satisfies the "uniform parabolicity" condition (see, for example, [1]). On other hand, it is easy to verify that the aforementioned change of variables conserves the spaces L^2 and $\mathcal{H}^{1,2}$. Consequently, we have proved the following theorem:

Theorem 3.2.1. *Problem (3.5),(3.6) admits a (unique) solution $u \in \mathcal{H}^{1,2}(D_1)$.*

3.3 Well-posedeness results for Problem (3.1),(3.2) in a domain which can be transformed into a half strip

In this case, we set

$$D_2 := \left\{(t,x,y) \in \mathbb{R}^3 : t > 0;\ 0 < \sqrt{x^2+y^2} < \varphi(t)\right\}$$

with $\varphi(0) > 0$ and consider the following problem: to find a function $u \in \mathcal{H}^{1,2}(D_2)$ that satisfies the equation

$$\partial_t u - c(t)(\partial_x^2 u + \partial_y^2 u) = f \ \text{ a.e. on } D_2 \tag{3.11}$$

and the boundary condition

$$u|_{\partial D_2} = 0, \tag{3.12}$$

where $f \in L^2(D_2)$.

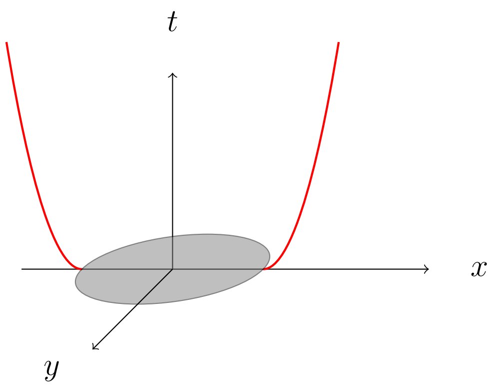

Fig.3: The unbounded domain D_2.

The change of variables indicated in the previous section transforms D_2 into the half strip $P =]0, +\infty[\times D(0,1)$, where $D(0,1)$ is the unit disk of $\mathbb{R}^2$. So Problem (3.11)-(3.12) can be written as follows: to find a function $u \in \mathcal{H}^{1,2}(P)$ that satisfies the equation

$$\partial_t u + (a(t,x)\,\partial_x u + a(t,y)\,\partial_y u) - b(t)(\partial_x^2 u + \partial_y^2 u) = f \text{ a.e. on } P \tag{3.13}$$

and the boundary condition

$$u|_{\partial P} = 0, \tag{3.14}$$

where the coefficients $a(t,x)$, $a(t,y)$ and $b(t)$ are those defined in Section 2. Let f_n be the restriction $f|_{]0,n[\times D(0,1)}$ for all $n \in \mathbb{N}^*$. Then, Theorem 3.2.1 shows that for all $n \in \mathbb{N}^*$, there exists a function $u_n \in \mathcal{H}^{1,2}(P_n)$ which solves the problem

$$\begin{cases} \partial_t u_n + (a(t,x)\,\partial_x u_n + a(t,y)\,\partial_y u_n) - b(t)(\partial_x^2 u_n + \partial_y^2 u_n) = f_n \in L^2(P_n), \\ u_n|_{\partial P_n \setminus (\{n\} \times D(0,1))} = 0, \end{cases} \tag{3.15}$$

where $P_n =]0, n[\times D(0,1)$. Now, let us prove an "energy" type estimate for the solutions u_n which will allow us to solve Problem (3.15) and then equivalently Problem (3.11)-(3.12).

Proposition 3.3.1. *There exists a constant K independent of n such that*

$$\|u_n\|_{\mathcal{H}^{1,2}(P_n)} \leq K \|f\|_{L^2(P)}.$$

In order to prove Proposition 3.3.1, we need the following lemmas:

Lemma 3.3.1. *There exists a constant $C > 0$ independent of n such that*

$$\left\|\partial_{xy}^2 u_n\right\|_{L^2(P_n)}^2 + \left\|\partial_x^2 u_n\right\|_{L^2(P_n)}^2 + \left\|\partial_y^2 u_n\right\|_{L^2(P_n)}^2 \leq C \|\Delta u_n\|_{L^2(P_n)}^2.$$

The previous lemma is a consequence of Lemma 1.2.1 and Grisvard-Looss [9, Theorem 2.2].

Lemma 3.3.2. *There exists a constant K independent of n such that*

$$\|u_n\|_{L^2(P_n)} \leq K \|\nabla u_n\|_{L^2(P_n)} \leq K \|f\|_{L^2(P)},$$

where $\nabla u_n = (\partial_x u_n, \partial_y u_n)$.

Proof. The Poincaré inequality gives $\|u_n\|_{L^2(P_n)} \leq \|\nabla u_n\|_{L^2(P_n)}$. Moreover, by developing the inner product $\langle \partial_t u_n + (a(t,x)\partial_x u_n + a(t,y)\partial_y u_n) - b(t)(\partial_x^2 u_n + \partial_y^2 u_n), u_n \rangle$ in $L^2(P_n)$, we obtain

$$\begin{array}{rcl}\langle f_n, u_n \rangle & = & \int_{P_n} u_n[\partial_t u_n + (a(t,x)\partial_x u_n + a(t,y)\partial_y u_n) - b(t)(\partial_x^2 u_n + \partial_y^2 u_n)]dtdxdy \\ & = & \frac{1}{2}\int_{\Gamma_n} |u_n|^2 dxdy + \int_{P_n} \frac{\varphi'(t)}{\varphi(t)} u_n^2(t,x,y)\, dtdxdy + \int_{P_n} b(t)|\nabla u_n|^2 dtdxdy.\end{array}$$

where $\Gamma_n = \left\{(n,x,y) \in \mathbb{R}^3 : 0 < \sqrt{x^2+y^2} < 1\right\}$. Since the function φ increases, we obtain

$$\langle f_n, u_n \rangle \geq \int_{P_n} b(t)|\nabla u_n|^2 dtdxdy \geq C\|\nabla u_n\|^2_{L^2(P_n)}.$$

Hence, for all $\epsilon > 0$,

$$\begin{array}{rcl}\|\nabla u_n\|^2_{L^2(P_n)} & \leq & \frac{1}{C}\|u_n\|_{L^2(P_n)}\|f_n\|_{L^2(P_n)} \\ & \leq & \frac{1}{C\epsilon}\|f\|^2_{L^2(P)} + \frac{\epsilon}{C}\|u_n\|^2_{L^2(P_n)}.\end{array}$$

By using the Poincaré inequality, we obtain

$$(1-\frac{\epsilon}{C})\|\nabla u_n\|^2_{L^2(P_n)} \leq \frac{1}{C\epsilon}\|f\|^2_{L^2(P)}.$$

Choosing ϵ small enough in the previous inequality, we prove the existence of a constant K such that

$$\|\nabla u_n\|_{L^2(P_n)} \leq K\|f\|_{L^2(P)}.$$

□

Remark 3.3.1. *Similar computations show that the same result holds true when we substitute the condition that φ increases in a neighborhood of $+\infty$ by the following:*

$$|\varphi'(t)|\varphi(t) \leq Mc(t).$$

Proof of Proposition 3.3.1: Set

$$Lu_n = \partial_t u_n + (a(t,x)\partial_x u_n + a(t,y)\partial_y u_n) - b(t)\Delta u_n,$$

where $\Delta u_n = \partial_x^2 u_n + \partial_y^2 u_n$. We have

$$\begin{array}{rcl}\|f_n\|^2_{L^2(P_n)} & = & \langle Lu_n, Lu_n \rangle \\ & = & \|\partial_t u_n\|^2_{L^2(P_n)} + \|a(t,x)\partial_x u_n\|^2_{L^2(P_n)} + \|a(t,y)\partial_y u_n\|^2_{L^2(P_n)} + \|b(t)\Delta u_n\|^2_{L^2(P_n)} \\ & & +2\int_{P_n} a(t,x)\partial_t u_n \partial_x u_n dtdxdy + 2\int_{P_n} a(t,y)\partial_t u_n \partial_y u_n dtdxdy \\ & & -2\int_{P_n} a(t,x)b(t)\partial_x u_n \Delta u_n dtdxdy - 2\int_{P_n} a(t,y)b(t)\partial_y u_n \Delta u_n dtdxdy \\ & & -2\int_{P_n} b\partial_t u_n \Delta u_n dtdxdy.\end{array}$$

Observe that the coefficients $a(t,x)$, $a(t,y)$ and $b(t)$ are bounded. So, thanks to Lemma 3.3.2, for all $\epsilon > 0$ we obtain

$$
\begin{aligned}
& \|\partial_t u_n\|^2_{L^2(P_n)} + \|b(t)\,\Delta u_n\|^2_{L^2(P_n)} - 2\textstyle\int_{P_n} b\partial_t u_n \Delta u_n dtdxdy \\
\leq\ & \|f\|^2_{L^2(P)} + \|a(t,x)\,\partial_x u_n\|^2_{L^2(P_n)} + \|a(t,y)\,\partial_y u_n\|^2_{L^2(P_n)} + 2\,\|\partial_t u_n\|_{L^2(P_n)}\,\|a(t,x)\partial_x u_n\|_{L^2(P_n)} \\
& +2\,\|\partial_t u_n\|_{L^2(P_n)}\,\|a(t,y)\partial_y u_n\|_{L^2(P_n)} + 2\,\|\Delta u_n\|_{L^2(P_n)}\,\|a(t,x)b(t)\partial_x u_n\|_{L^2(P_n)} \\
& +2\,\|\Delta u_n\|_{L^2(P_n)}\,\|a(t,y)b(t)\partial_y u_n\|_{L^2(P_n)} \\
\leq\ & \|f\|^2_{L^2(P)} + K_1\left(1+\tfrac{2}{\epsilon}\right)[\|\partial_x u_n\|^2_{L^2(P_n)} + \|\partial_y u_n\|^2_{L^2(P_n)}] + 2\epsilon\,\|\partial_t u_n\|^2_{L^2(P_n)} + 2\epsilon\,\|b(t)\Delta u_n\|_{L^2(P_n)} \\
\leq\ & K_\epsilon\,\|f\|^2_{L^2(P)} + 2\epsilon\,\|\partial_t u_n\|^2_{L^2(P_n)} + 2\epsilon\,\|\Delta u_n\|^2_{L^2(P_n)}\,,
\end{aligned}
$$

where K_ϵ and K_1 are constants independent of n. Consequently

$$
(1-2\epsilon)\left(\|\partial_t u_n\|^2_{L^2(P_n)} + \|b(t)\Delta u_n\|^2_{L^2(P_n)}\right) \leq 2\int_{P_n} b(t)\partial_t u_n \Delta u_n dtdxdy + K_\epsilon\,\|f\|^2_{L^2(P)}\,. \tag{3.16}
$$

Estimation of $2\int_{P_n} b(t)\partial_t u_n \Delta u_n dtdxdy$: We have

$$
\partial_t u_n.\Delta u_n = \partial_x\left(\partial_t u_n \partial_x u_n\right) + \partial_y\left(\partial_t u_n \partial_y u_n\right) - \frac{1}{2}[\partial_t\left(\partial_x u_n\right)^2 + \partial_t\left(\partial_y u_n\right)^2].
$$

Then

$$
\begin{aligned}
2\textstyle\int_{P_n} b(t)\partial_t u_n \Delta u_n dtdxdy &= 2\textstyle\int_{P_n} b(t)\partial_x\left(\partial_t u_n \partial_x u_n\right)dtdxdy + 2\textstyle\int_{P_n} b(t)\partial_y\left(\partial_t u_n \partial_y u_n\right)dtdxdy \\
&\quad - \textstyle\int_{P_n} b(t)\partial_t\left(\partial_x u_n\right)^2 dtdxdy - \textstyle\int_{P_n} b(t)\partial_t\left(\partial_y u_n\right)^2 dtdxdy \\
&= \textstyle\int_{\partial P_n} b(t)[-\left|\nabla u_n\right|^2 \nu_t + 2\partial_t u_n(\partial_x u_n \nu_x + \partial_y u_n \nu_y)]d\sigma \\
&\quad + \textstyle\int_{P_n} b'(t)\left|\nabla u_n\right|^2 dtdxdy,
\end{aligned}
$$

where ν_t, ν_x and ν_y are the components of the unit outward normal vector at ∂P_n. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of P_n where $t=0$ and $\sqrt{x^2+y^2}=1$, we have $u_n=0$ and consequently $\partial_x u_n = \partial_y u_n = 0$. The corresponding boundary integral vanishes. On Γ_n, the part of the boundary where $t=n$, we have $\nu_x=\nu_y=0$ and $\nu_t=1$. Accordingly the corresponding boundary integral

$$
-\int_{\Gamma_n} b(n)\left|\nabla u_n\right|^2(n,x,y)\,dxdy
$$

is negative. Finally,

$$2\int_{P_n} b(t)\partial_t u_n \Delta u_n dtdxdy \quad = \quad -\int_{\Gamma_n} b(n) \left|\nabla u_n\right|^2 (n,x,y)\, dxdy + \int_{P_n} b'(t) \left|\nabla u_n\right|^2 dtdxdy.$$

Note that the functions $b(t)$ and $b'(t)$ are bounded. So, by using Lemma 3.3.2, we deduce

$$\begin{aligned} 2\int_{P_n} b(t)\partial_t u_n \Delta u_n dtdxdy &\leq \int_{P_n} b'(t) \left|\nabla u_n\right|^2 dtdxdy \\ &\leq K_2 \left\|\nabla u_n\right\|^2_{L^2(P_n)} \\ &\leq K_3 \left\|f\right\|^2_{L^2(P)}, \end{aligned}$$

where K_2 and K_3 are constants independent of n. Consequently, Choosing $\epsilon = \frac{1}{4}$ in the relationship (3.16), we obtain

$$\left\|\partial_t u_n\right\|^2_{L^2(P_n)} + \left\|\Delta u_n\right\|^2_{L^2(P_n)} \leq K \left\|f\right\|^2_{L^2(P)}.$$

This ends the proof of Proposition 3.3.1.

Remark 3.3.2. *We obtain the solution u of Problem (3.11)-(3.12) by letting n go to infinity in the previous proposition. The uniqueness can be proved as in Theorem 3.2.1.*

Finally, we have proved the following Theorem:

Theorem 3.3.1. *Problem (3.11)-(3.12) admits a (unique) solution $u \in \mathcal{H}^{1,2}(D_2)$.*

3.4 Local in time result

Let T be a small enough positive real number. We set

$$D_3 := \left\{(t,x,y) \in \mathbb{R}^3 : 0 < t < T;\ 0 \leq \sqrt{x^2+y^2} < \varphi(t)\right\}$$

with $\varphi(0) = 0$ and consider the following problem: to find a function $u \in \mathcal{H}^{1,2}(D_3)$ that satisfies the equation

$$\partial_t u - c(t)(\partial_x^2 u + \partial_y^2 u) = f \text{ a.e. on } D_3 \tag{3.17}$$

and the boundary condition

$$u|_{\partial D_3 \setminus \Gamma_T} = 0, \tag{3.18}$$

where $f \in L^2(D_3)$ and Γ_T is the part of ∂D_3 where $t = T$, defined by

$$\Gamma_T := \left\{ (T, x, y) \in \mathbb{R}^3 : 0 < \sqrt{x^2 + y^2} < \varphi(T) \right\}.$$

Set

$$Q_n = \left\{ (t, x, y) \in D_3 : \frac{1}{n} < t < T \right\}, \; n \in \mathbb{N}^* \text{ and } \frac{1}{n} < T.$$

For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, we set $f_n = f|_{Q_n} \in L^2(Q_n)$, and denote by $u_n \in \mathcal{H}^{1,2}(Q_n)$ the solution of the following problem:

$$\begin{cases} \partial_t u_n - c(t)(\partial_x^2 u_n + \partial_y^2 u_n) = f_n \text{ a.e. on } Q_n, \\ u_n|_{\partial Q_n \setminus \Gamma_T} = 0. \end{cases} \tag{3.19}$$

Such a solution exists by Theorem 3.2.1.

Proposition 3.4.1. *There exists a constant $K > 0$ independent of n such that*

$$\|u_n\|^2_{\mathcal{H}^{1,2}(Q_n)} \leq K \|f_n\|^2_{L^2(Q_n)} \leq K \|f\|^2_{L^2(D_3)}.$$

Remark 3.4.1. *Let $\epsilon > 0$ be a real which we will choose small enough. The hypothesis (3.3) implies the existence of a real number $T > 0$ small enough such that*

$$|\varphi'(t)\varphi(t)| \leq \epsilon, \text{ for all } t \in (0, T). \tag{3.20}$$

In order to prove Proposition 3.4.1, we need some preliminary results.

Lemma 3.4.1. *For a fixed $t \in \left]\frac{1}{n}, T\right[$, let Ω_t be the bounded domain of $\mathbb{R}^2$ defined by*

$$\Omega_t = \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq \sqrt{x^2 + y^2} < \varphi(t) \right\}.$$

Then, there exists a constant $C > 0$ such that

(a) $\max(\|\partial_x u_n\|^2_{L^2(\Omega_t)}, \|\partial_y u_n\|^2_{L^2(\Omega_t)}) \leq C\varphi^2(t) \|\Delta u_n\|^2_{L^2(\Omega_t)}$,

(b) $\|u_n\|^2_{L^2(\Omega_t)} \leq C\varphi^4(t) \|\Delta u_n\|^2_{L^2(\Omega_t)}$.

Proof. It is a direct consequence of Lemma 1.2.1. Indeed, let $t \in \left]\frac{1}{n}, T\right[$ and define the following change of variables

$$\begin{array}{ccc} D(0,1) & \to & \Omega_t \\ (x, y) & \longmapsto & (\varphi(t)x, \varphi(t)y) = (x', y'). \end{array}$$

Set

$$v(x,y) = u_n(\varphi(t)x, \varphi(t)y),$$

(a) We have

$$\begin{aligned}
\|\partial_x v\|^2_{L^2(D(0,1))} &= \int_{D(0,1)} (\partial_x v)^2 (x,y)\, dxdy \\
&= \int_{\Omega_t} (\partial_{x'} u_n)^2 (x',y')\, \varphi^2(t) \tfrac{1}{\varphi^2(t)} dx'dy' \\
&= \int_{\Omega_t} (\partial_{x'} u_n)^2 (x',y')\, dx'dy' \\
&= \|\partial_{x'} u_n\|^2_{L^2(\Omega_t)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|\Delta v\|^2_{L^2(D(0,1))} &= \int_{D(0,1)} \left[\left(\partial_x^2 v + \partial_y^2 v\right)(x,y)\right]^2 dxdy \\
&= \int_{\Omega_t} \left[\varphi^2(t)\left(\partial_{x'}^2 u_n + \partial_{y'}^2 u_n\right)\right]^2 (x',y') \tfrac{1}{\varphi^2(t)} dx'dy' \\
&= \varphi^2(t) \left[\partial_{x'}^2 u_n + \partial_{y'}^2 u_n\right]^2 (x',y')\, dx'dy' \\
&= \varphi^2(t) \|\Delta u_n\|^2_{L^2(\Omega_t)}.
\end{aligned}$$

Using the inequality

$$\|\partial_x v\|^2_{L^2(D(0,1))} \leq C \|\Delta v\|^2_{L^2(D(0,1))}$$

of Lemma 1.2.1, we obtain the desired inequality

$$\|\partial_{x'} u_n\|^2_{L^2(\Omega_t)} \leq C\varphi^2(t) \|\Delta u_n\|^2_{L^2(\Omega_t)}.$$

By a similar argument, we get

$$\|\partial_{y'} u_n\|^2_{L^2(\Omega_t)} \leq C\varphi^2(t) \|\Delta u_n\|^2_{L^2(\Omega_t)}.$$

(b) We have

$$\|v\|^2_{L^2(D(0,1))} = \varphi^{-2}(t) \|u_n\|^2_{L^2(\Omega_t)}.$$

On the other hand,

$$\|\Delta v\|^2_{L^2(D(0,1))} = \varphi^2(t) \|\Delta u_n\|^2_{L^2(\Omega_t)}.$$

Using the inequality

$$\|v\|^2_{L^2(D(0,1))} \leq C \|\Delta v\|^2_{L^2(D(0,1))}$$

of Lemma 1.2.1, we obtain the desired inequality

$$\|u_n\|^2_{L^2(\Omega_t)} \leq C\varphi^4(t) \|\Delta u_n\|^2_{L^2(\Omega_t)}.$$

□

Proof of Proposition 3.4.1: This result can be obtained by following step by step the proof of [[13], Proposition 3.2].

Theorem 3.4.1. *Problem (3.17)-(3.18) admits a (unique) solution $u \in \mathcal{H}^{1,2}(D_3)$.*

Proof. We obtain the solution u of Problem (3.17)-(3.18) by letting n go to infinity in the previous proposition. $\square$

3.5 Main result

The proof of Theorem 3.1.1 can be obtained by subdividing the domain

$$D = \left\{(t, x, y) \in \mathbb{R}^3 : t > 0;\ 0 \leq \sqrt{x^2 + y^2} < \varphi(t)\right\}$$

into three subdomains D_1, D_2 and D_3. We set $D = D_1 \cup D_2 \cup D_3 \cup \Gamma_{T_1} \cup \Gamma_{T_2}$, where

$$D_1 = \{(t, x, y) \in D :\ 0 < t < T_1\},\ D_2 = \{(t, x, y) \in D :\ T_1 < t < T_2\},$$

$$D_3 = \{(t, x, y) \in D :\ t > T_2\},\ \Gamma_{T_1} := \left\{(T_1, x, y) \in \mathbb{R}^3 : 0 < \sqrt{x^2 + y^2} < \varphi(T_1)\right\},$$

and

$$\Gamma_{T_2} := \left\{(T_2, x, y) \in \mathbb{R}^3 : 0 < \sqrt{x^2 + y^2} < \varphi(T_2)\right\},$$

with T_1 is a small enough positive number and T_2 is an arbitrary positive number such that $T_2 > T_1$. In the sequel f stands for an arbitrary fixed element of $L^2(D)$ and $f^{(i)} = f_{|D_i}$, $i = 1, 2, 3$.

In the conical domain D_1, there exists a unique solution $u_1 \in \mathcal{H}^{1,2}(D_1)$ of the problem

$$\begin{cases} \partial_t u_1 - c(t)[\partial_x^2 u_1 + \partial_y^2 u_1] = f^{(1)} \in L^2(D_1), \\ u_1\big|_{\partial D_1 - \Gamma_{T_1}} = 0, \end{cases} \tag{3.21}$$

Hereafter, we denote the trace $u_1\big|_{\Gamma_{T_1}}$ by ψ_1 which is in the Sobolev space $H^1(\Gamma_{T_1})$ because $u_1 \in H^{1,2}(D_1)$ (see [21]). Now, consider the following problem in D_2

$$\begin{cases} \partial_t u_2 - c(t)[\partial_x^2 u_2 + \partial_y^2 u_2] = f^{(2)} \in L^2(D_2), \\ u_2\big|_{\Gamma_{T_1}} = \psi_1, \\ u_2\big|_{\partial D_2 - (\Gamma_{T_1} \cup \Gamma_{T_2})} = 0. \end{cases} \tag{3.22}$$

We use the following result, which is a consequence of Theorem 4.3 ([21], Vol.2) to solve Problem (3.22).

Proposition 3.5.1. *Let Q be the cylinder $]0,T[\times D(0,1)$ where $D(0,1)$ is the unit disk of $\mathbb{R}^2$, $f \in L^2(Q)$ and $\phi \in H^1(\gamma_0)$. Then, the problem*

$$\begin{cases} \partial_t u - c(t)\Delta u = f \text{ in } Q, \\ u\big|_{\gamma_0} = \phi, \\ u\big|_{\gamma_0 \cup \gamma_1} = 0, \end{cases}$$

where $\gamma_0 = \{0\} \times D(0,1)$, $\gamma_1 =]0,T[\times \partial D(0,1)$, admits a (unique) solution $u \in H^{1,2}(Q)$.

Thanks to the transformation

$$(t,x,y) \longmapsto (t,x',y') = (t, \varphi(t)\, x, \varphi(t)\, y),$$

we deduce the following result :

Proposition 3.5.2. *Problem* (3.22) *admits a (unique) solution $u_2 \in \mathcal{H}^{1,2}(D_2)$.*

Hereafter, we denote the trace $u_2|_{\Gamma_{T_2}}$ by ψ_2 which is in the Sobolev space $H^1(\Gamma_{T_2})$ because $u_2 \in \mathcal{H}^{1,2}(D_2)$. Now, consider the following problem

$$\begin{cases} \partial_t u_3 - c(t)[\partial_x^2 u_3 + \partial_y^2 u_3] = f^{(3)} \in L^2(D_3), \\ u_3\big|_{\Gamma_{T_2}} = \psi_2, \\ u_3\big|_{\partial D_3 - \Gamma_{T_2}} = 0. \end{cases} \tag{3.23}$$

By similar arguments like those used previously, we deduce the following result :

Proposition 3.5.3. *Problem* (3.23) *admits a (unique) solution $u_3 \in \mathcal{H}^{1,2}(D_3)$.*

Finally, the function u defined by 1

$$u := \begin{cases} u_1 & \text{in} \quad D_1, \\ u_2 & \text{in} \quad D_2, \\ u_3 & \text{in} \quad D_3, \end{cases}$$

is the (unique) solution of our problem. This ends the proof of Theorem 3.1.1.

CONCLUSION AND PROSPECTS

There are many natural systems which evolve on time-dependent spatial domains. In this thesis, we have studied two boundary value problems associated to the following parabolic equation

$$\partial_t u - c(t)\Delta u = f,$$

where Δ is the Laplacian operator, $c(t)$ is a time dependent coefficient and $f \in L^2$. The equation is posed in unbounded non regular domains and it is associated to different boundary conditions.

Whereas a comprehensive theory of such kind of problems exists in the case of bounded regular domains, the literature in the case of unbounded non-smooth domains does not seem to be very rich. One of the central results of this theory is the so-called "schift theorem": "If the second member is an element of a certain given regularity (in a Sobolev space for example), then the solution admits this regularity + 2." In this thesis, we have given sufficient conditions on the functions of parametrization of the domains, and on the coefficients in order to bring to life this result when the domains are unbounded and non-smooth and the second member is in the Lebesgue space of square integrable functions, L^2.

First, we have proved well-posedness and regularity results for the above mentioned parabolic equation, subject to Dirichlet-Robin type boundary conditions and posed in an unbounded in time non-rectangular domain. In the second work, the two dimensional case with Cauchy-Dirichlet boundary conditions is studied in the case of unbounded conical domains.

In forthcoming works, the results obtained in this thesis will be extended at least in the following directions:

1. The function f on the right-hand side of the above mentioned equation, may be taken in L^p, where $p \in]1, +\infty[$ or in Holder spaces. The difficulty with the space L^p, $p \neq 2$, is that this space is not a Hilbert space, and so there is no an inner product. So, the domain decomposition method used in this cannot be generalized in this sense.

2. Consider higher-order parabolic equations and other boundary conditions.

BIBLIOGRAPHY

[1] Yu. A. Alkhutov and I. T. Mamedov, *The first boundary value problem for nondivergence secondorder parabolic equations with discontinuous coefficients,* Russian Acad. Sc. Sbornik Math. 59(2)(1988), 471–495.

[2] Yu. A. Alkhutov, *L_p-estimates for the solution of the Dirichlet problem for the heat equation in a ball,* (in Russian), Sovrem. Mat. Prilozh., No. 29 (2005), 3–13; translation in J. Math. Sci. (N. Y.) 142 (3) (2007), 2021-2032.

[3] V. N. Aref'ev, L. A. Bagirov, *Solutions of the heat equation in domains with singularities,* (in Russian), *Mat. Zametki* **64** (1998), no. 2, 163–179; translation in Math. Notes 64, no. 1-2, (1998), 139-153.

[4] Tahir Boudjeriou, Arezki Kheloufi, *Global in space regularity results for the heat equation with Robin–Neumann type boundary conditions in time-varying domains,* J. Sib. Fed. Univ. Math. Phys., 2019, Volume 12, Issue 3, 355-370.

[5] H. Brezis, *Analyse Fonctionnelle at Applications,* Masson, 1983.

[6] E. J. Crampin, E. A. Gaffney, P. K. Maini, *Reaction and diffusion on growing domains: scenarios for robust pattern formation,* Bull. Math. Biol. 61, No. 6, (1999), 1093-1120.

[7] S.P. Degtyarev, *The solvability of the first initial-boundary problem for parabolic and degenerate parabolic equations in domains with a conical point,* Sbornik Math. 201, 7 (2010) 999–1028.

[8] L. C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Americain Mathematical Society, 19, 1998.

[9] P. Grisvard, G. Looss, Problèmes aux limites unilatéraux dans des domaines non réguliers, Journées Equations aux Dérivées Partielles, (1976) 1-26.

[10] S. Guesmia, *Large time and space size behaviour of the heat equation in non-cylindrical domains,* Arch. Math. 101 (2013), 293-299.

[11] S. Hofmann, J. L. Lewis, *The L^p Neumann problem for the heat equation in non-cylindrical domains,* J. Funct. Anal., 220 (2005), no. 1, 1-54.

[12] A. Kheloufi, B.-K. Sadallah, On the regularity of the heat equation solution in non-cylindrical domains: two approaches, Appl. Math. Comput. 218, No. 5, (2011), 1623-1633.

[13] A. Kheloufi, B.-K. Sadallah, Study of the heat equation in a symmetric conical type domain of $\mathbb{R}^{N+1}$, Math. Methods Appl. Sci. 37, No. 12, 1807-1818 (2014).

[14] A. Kheloufi, B. K. Sadallah, *Study of a parabolic equation with mixed Dirichlet-Neumann type boundary conditions in unbounded noncylindrical domains,* Journal of Advanced Research in Applied Mathematics, Vol. 7, Issue. 4, 2015, 62-77.

[15] A. Kheloufi, *On parabolic equations with mixed Dirichlet-Robin type boundary conditions in a non-rectangular domain,* Mediterr. J. Math. 13 (2016), 1787-1805.

[16] E. Knobloch, R. Krechetnikov, *Stability on time-dependent domains,* J. Nonlinear Sci. 24, No. 3, (2014) 493-523.

[17] V. A. Kondrat'ev, *Boundary problems for parabolic equations in closed regions,* Am. Math. Soc. Providence. R I. (1966), 450-504.

[18] V. A. Kozlov and V. G.Maz'ya, *On singularities of a solution to the first boundary value problem for the heat equation in domains with conical points, I,* Izv. Vyssh. Uchebn. Zaved., Ser. Mat., 2 (1987), 38–46

[19] V. A. Kozlov and V. G.Maz'ya, *On singularities of a solution to the first boundary value problem for the heat equation in domains with conical points II,* Izv. Vyssh. Uchebn. Zaved., Ser. Mat., 3 (1987), 37–44.

[20] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type,* (Russian), Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I., 1968.

[21] J. L. Lions, E. Magenes, *Problèmes aux Limites non Homogènes et Applications,* $1, 2$, Dunod, Paris (1968).

[22] F. Paronetto, *An existence result for evolution equations in non-cylindrical domains,* Nonlinear Differ. Equ. Appl. 20 (2013), 1723-1740.

[23] A. I. Nazarov, *L^p-estimates for a solution to the Dirichlet problem and to the Neumann problem for the heat equation in a wedge with edge of arbitrary codimension,* J. of Math. Sci., 106, 3 (2001), 2989-3014.

[24] M. A. Ragusa, *Cauchy-Dirichlet problem associated to divergence form parabolic equations,* Communications in Contemporary Mathematics, vol. 6, no. 3, (2004), 377-393.

[25] B.-K. Sadallah, *Etude d'un problème $2m-$parabolique dans des domaines plan non rectangulaires,* Boll. Un. Mat. Ital., $(5), 2-B$ (1983), $51-112$.

[26] B. K. Sadallah, *Regularity of a parabolic equation solution in a non-smooth and unbounded domain,* J. Aust. Math. Soc., 84 (2)(2008), 265-276.

ABSTRACT

This thesis is designed to give some results of existence and uniqueness of solutions for someparabolic equations posed in unbounded in time non-cylindrical domains. We give sufficient conditions on the functions of the parametrization of the non regular domains and on the coefficients of the equations under which our problems admit unique solutions. We study the global regularity problem in a suitable parabolic Sobolev space. The method used to prove our main results is based on the technique of the decomposition of domains.

Key words. parabolic equations, heat equation, non-rectangular domains, conical domains, unbounded domains, Dirichlet-Robin conditions, anisotropic Sobolev spaces.

RÉSUMÉ

Cette thèse a pour but de donner des résultats d'existence et d'unicité de solutions pour certaines équations paraboliques posées dans des domaines non cylindriques et non bornés en temps. Nous donnons des conditions suffisantes sur les fonctions de paramétrisation des domaines non réguliers et sur les coefficients des équations sous lesquelles nos problèmes admettent une solution unique. Nous étudions le problème de régularité globale dans un espace de Sobolev parabolique approprié. La méthode utilisée pour démontrer nos principaux résultats est basée sur la technique de la décomposition de domaines.

Mots clés. Equations paraboliques, équation de la chaleur, domaines non rectangulaires, domaines coniques, domaines non bornés, conditions de Dirichlet-Robin, espaces de Sobolev anisotropes.

الملخص

تهدف هذه الأطروحة الى تقديم نتائج وجود ووحدانية الحلول لبعض المعادلات المكافئية المطروحة في الساحات غير الأسطوانية غير المحدودة بالنسبة للزمن. نقدم شروطًا كافية على التوابع المعلمة للساحات غير الصقيلة وعلى معاملات المعادلات التي تحت ظروفها تقبل مسائلنا حلولًا وحدانية. ندرس مسالة الصقالة الكلية في فضاءات سوبوليف غير متناظرة المناسبة. الطريقة المستخدمة لإثبات نتائجنا الرئيسية مبنية على تقنية تجزئة الساحات

الكلمات الرئيسية. المعادلات المكافئية، معادلة الحرارة، الساحات غير المستطيلة، الساحات المخروطية، الساحات غير المحدودة، شروط ديريشليه-روبن، فضاءات سوبوليف غير المتناظرة