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THÈSE

EN VUE DE L'OBTENTION DU DIPLOME DE DOCTORAT

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Filière :Mathématiques
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Présentée par

BOUCHAL Lydia

Thème

Résultats d'existence pour des problèmes non linéaires

Soutenue le : 13/ 01 /2024 **Devant le Jury composé de :**

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A. MIRA University of Bejaia
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Department of Mathematics

Laboratory of Applied Mathematics (LMA)

THESIS

IN ORDER TO GET A DOCTORAL DEGREE

Domain: Mathematics and Informatics **Tracks:** Mathematics
Speciality: Analysis

Presented by
BOUCHAL Lydia

Theme

Existence results for nonlinear problems

Supported on: 13/01/2024 **In front of a jury composed of:**

Last and First names	Grade		
Ms. BOURAINE-BERDJOUDJ Louiza	Prof.	Univ. Bejaia	President
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Dedication

This work is dedicated:

To the candle of my life, my dear mother, and my dear father,

To my sisters, to my brothers,

To my supervisor Pr.KHELOUFI-MEBARKI Karima,

To all my fixed points.

Presentation

This thesis is focused on some questions related to existence, multiplicity and localization of positive solutions for some nonlinear mathematical problems that can be represented by equations of the form $Tx + Sx = x$ with x is on an appropriate closed convex set. The method used here is the fixed point theory on retracts of Banach spaces. More precisely, for proving the desired results we rely on the fixed point index theory for the sum of two operators. This work contains two principal types of results:

- Firstly, we have developed fixed point theorems for the sum $T + S$, by taking this sum such that S is a k-set contraction and $I - T$ is Lipschitz invertible. The obtained results were established by using a new topological approach of the fixed point index.

- Secondly, we have used the obtained fixed point theorems and other recent ones existing in the literature to discuss the existence, multiplicity, localization and positivity of solutions of divers kinds of boundary value problems for difference and differential equations. Most of the obtained existence results are illustrated by numerical examples.

This manuscript contains three essential parts: The first part "General concepts", contains the indispensable elements which will be needed throughout this thesis. In the second part "Difference equations", we gave firstly remainder of some concepts on linear difference equations and then we presented our contributions. In the third part "Differential equations", we presented our results on differential equations associated to boundary value conditions.

Contents

List of contributions

i. List of publications

- L. Bouchal and K. Mebarki, New multiple fixed point theorems for sum of two operators and application to a singular generalized Strum-Liouville multipoint BVP, Matematicki Vesnik, DOI: 10.57016/MV-38FENT80 (Article in press), 2023.
- L. Bouchal, K. Mebarki and S.G. Georgiev, Positive solutions for a class of non-autonomous second order difference equations via a new functional fixed point theorem. Archivum Mathematicum, 58(4):199-211, 2022.
- L. Bouchal, K. Mebarki and S.G. Georgiev, Existence of solutions for boundary value problems for first order impulsive difference equations, Boletim da Sociedade Paranaense de Matemática, DOI: 10.5269/bspm.62844 (Article in press), 2022.
- L. Bouchal and K. Mebarki, Fixed point theorem on functional intervals for sum of two operators and application in ODEs, International Journal of Nonlinear Analysis and Applications, DOI: 10.22075/IJNAA.2023.28477.3910 (Article in press), 2023.
- L. Bouchal, K. Mebarki and S.G. Georgiev, Nonnegative solutions for a class of fourth order singular eigenvalue problems, Accepted in Studia Universitatis Babes-Bolyai Mathematica.

ii. List of communications

- International conference: Recent Developpement in Ordinary and Partial Differential Equations (22-26 may 2022, Bejaia University). "**Existence result for a non autonomous second order difference equations via a new fixed point theorem**".
- The 4th International Conference on Pure and Applied Mathematics (22-23 june, 2022, Van, Turkey)."**Multiple fixed point results for sum of operators and application**".
- The 2nd National Conference on Mathematics and its Applications (17-18 september 2022, Bordj Bou Arréridj University). "**Multiple positive solutions for a singular multipoint value problem via fixed point theory for the sum of operators**".
- National conference: Second Conference on Mathematics and Applications of Mathematics (28-29 september 2022, Jijel University). "**Positive solutions for impulsive Difference equations with nonlinear boundary conditions**".
- Second National Conference on Applied Mathematics and Didactics (13 may 2023, Constantine). "**On existence of solutions for a singular fourth order boundary value problem with parameter**".

List of symbols

Δ	Difference operator.
$\sum$	Summation operator (Antidifference).
$\Gamma(z)$	Gamma function $\Gamma(z) = \int\limits_0^\infty e^{-t} t^{z-1}\, dt$.
$\binom{n}{k}$	$\frac{n!}{k!(n-k)!}$.
$W(n)$	Casoratian.
δ_{ts}	Kronecker delta .
$\mathcal{P}$	Cone.
$\mathcal{P}^*$	$\mathcal{P} \setminus \{0\}$ punctured cone.
∂B	Boundary of the set B.
$\mathring{\mathcal{B}}$	Interior of the set B.
$i(f, U, X)$	Fixed point index of f over U with respect to X.
$B(x, r)$	Open ball with center x and radius r.
$conv(A)$	The convex hull of A.
I	The identity map.
$\mathbb{R}^+$	The set of all nonnegative real numbers.
$\mathbb{R}^n$	The n-dimentional Euclidean space.
$\mathcal{C}(X, Y)$	The set of all continuous functions from X to Y.
$\mathcal{C}_b(X, Y)$	The set of all bounded continuous functions from X to Y.
$\mathbb{N}_{n_0}$	$\{n > n_0, \quad n_0 \in \mathbb{N}\}$.
BVPs	Boundary value problems.

By positive solution of a BVP posed on I we mean a function $u : I \to \mathbb{R}$ such that $u(t) \geq 0$ on I and verifies the posed BVP.

General Introduction

The aim purpose of this thesis is to investigate the existence, multiplicity and localization of positive solutions of abstract equations of the form $Ax + Bx = x$ in a suitable subset of a Banach space, by developing new fixed point results. We are also interested by establishing the existence of positive solutions for boundary value problems associated to difference and differential equations. The arguments are based on the obtained fixed point theorems as well as on recent existing ones in the literature.

Fixed point theory, as one of the most dynamic areas, that still flourishing at present has fascinated many researchers for the powerful and fruitful tools it provides to nonlinear analysis, whether as fixed point theorems or in the form of topological tools. Fixed point theorems are used for proving the existence, multiplicity and uniqueness of solutions to various mathematical problems, such as integral equations, ordinary differential equations, partial differential equations, differences equations and variational inequalities.

In fact, long list of problems in analysis fall into the category of solving a fixed point problem of a suitable operator, among them one can encounters operators that can be split in the form $T = A + B$, where A is a contraction in some sense, and B is a compact map, however, in some cases A may not always yield a contraction operator but another type of operator, for instance, a non-expansive or expansive one. Therefore, neither the Banach fixed point theorem nor the Schauder fixed point theorem immediately apply in this case. For that reason it becomes desirable to develop fixed point theorems for such situations. One of the first results in this direction, which combines the Banach's and Schauder's fixed point theorems, is Krasnosel'skii's theorem established on 1958, see [58]. The author's motivation came from the observation that the inversion of a perturbed differential operator yields the sum of a contraction and a compact operator when he studied a paper of Schauder on elliptic partial differential equations [80]. Krasnosel'skii then established and proved that the sum of two operators $A + B$ has a fixed point in a nonempty closed convex subset Ω of a Banach space X, whenever the two operators $A, B : \Omega \to X$ satisfy

(a) A is a contraction;

(b) B is compact;

(c) $Ax + By \in \Omega, \quad \forall x, y \in \Omega.$

This is a captivating result, and it has a number of interesting applications in studying the existence of solutions, and it has been improved and generalized in diverse directions by modifying its hypothesis. In 1930, Kuratowski introduced his measure of noncompactness because of the lack of compactness in infinite dimensional Banach spaces and in order to determine how a set is not compact, see [61]. This new measure of noncompactness has been used by Darbo to introduce a new version of the contraction hypothesis. He calls A a k-set contraction mapping if $\alpha(A(D)) \leq k\alpha(D)$ for all bounded set D of Ω, where α denotes the Kuratowski measure of noncompactness and $k \in [0, 1[$. He then proved that any k-set contraction mapping which maps continuouisly Ω into itself, has at least a fixed point. In fact, since the sum of a contraction and a compact mapping turned out to be a k-set contraction, Darbo's fixed point theorem is considered as an extension of Krasnosel'skii's fixed point theorem. Since then many researchers have studied the existence of the solutions for the operational equation $Ax + Bx = x,\ x \in \Omega$, under different combinations of A, B and Ω. Some results in this direction can be seen in [18, 16, 43, 34, 40, 71, 88, 89].

In the investigation of certain problems arising from diverse areas of applied sciences only positive solutions that make sense, where this solution may describe a velocity, a density, and more. In a Banach space, the positivity condition can be described by a cone. Therefore, since the publication of the monograph 'Positive Solutions of Operator Equations' in the year 1964 by Krasnosel'skii, see [59], a lot of research articles on the theory of positive solutions of nonlinear problems have appeared.

Among the fixed point theorems established in ordered Banach spaces, we find the class of functional fixed point theorems used to prove existence, localization and multiplicity of positive solutions to boundary value problems. These theorems originate with Krasnosel'skii fixed point theorem in 1964, see [59] where the functional used was the norm and the fixed point is localized in a conical shell of the form $\{x \in \mathcal{P},\ a \leq \|x\| \text{ and } \|x\| \leq b\}$ for $0 < a < b$. This theorem has been improved by many authors, we find for instance the compression-expansion fixed point theorem of norm type, developed by Guo in [46, 48], stated as follows:

Let $\mathcal{P}$ be a cone of a Banach space X and assume that there exist two positive constants a; b with $a \neq b$, then a completely continuous map $F : \mathcal{P}_{a,b} \to \mathcal{P}$ has at least one fixed point in the conical shell

$$\mathcal{P}_{a,b} = \{x \in \mathcal{P},\ a \leq \|x\| \text{ and } \|x\| \leq b\} \text{ for } 0 < a < b,$$

under the following two conditions:

$$\|Fx\| \leq \|x\| \text{ for every } x \in \mathcal{P} \text{ with } \|x\| = a,$$

$$\|Fx\| \geq \|x\| \text{ for every } x \in \mathcal{P} \text{ with } \|x\| = b.$$

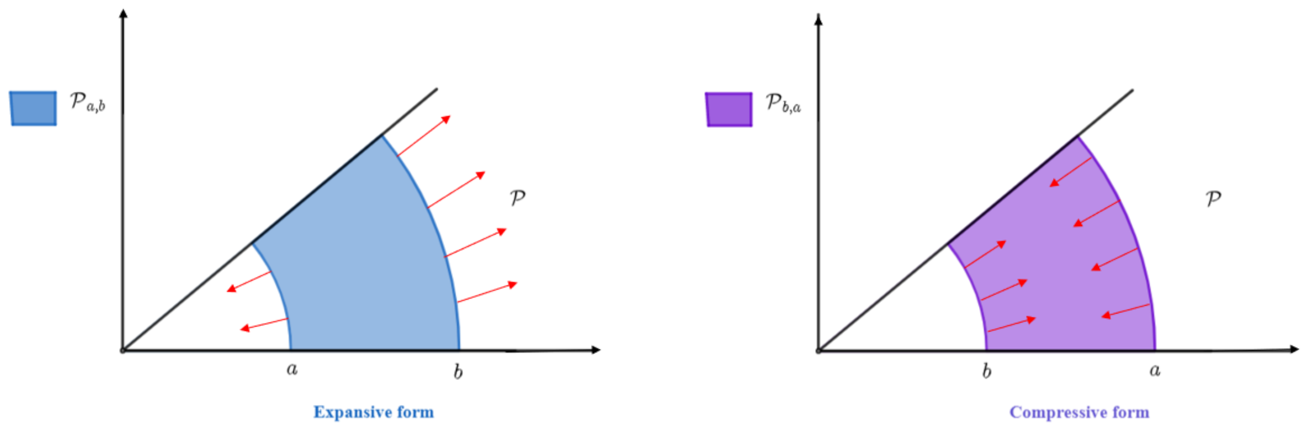

Figure 1: Illustration of cone compression-expansion fixed point theorem in case where X is the two-dimensional plane $\mathbb{R}^2$

In 1979, both Leggett and Williams, see [63], presented criteria that guarantee the existence of a fixed point for continuous, compact maps that did not require the operator to be invariant on the underlying sets, utilizing a norm in the upper boundary and replacing the norm used in the lower boundary by a positive concave functional such that $a \leq \alpha(x)$. In that sense, the Leggett-Williams fixed point theorem generalized the compression-expansion fixed point theorem of norm type and then fixed points are localized in sets of the form

$$\mathcal{P}(\alpha, a, b) = \{x \in \mathcal{P},\ a \leq \alpha(x) \text{ and } \|x\| \leq b\}.$$

Later, the Leggett-Williams fixed theorem was established with a little change (see [9, Theorem 16]) by introducing a convex functional β instead of the norm in the upper boundary, and then fixed points are localized in sets of the form

$$\mathcal{P}(\beta, \alpha, a, b) = \{x \in \mathcal{P},\ a \leq \alpha(x) \text{ and } \beta(x) \leq b\}.$$

Recently, the extensions of the Leggett-Williams theorem has attracted many researchers. In fact, using functionals we gain more flexibility and freedom to apply this kind of theorems in a wider variety of situations.

The fixed point index provides a useful tool for proving fixed point theorems. Nevertheless, in comparison to Leray-Schauder degree, the fixed point index is not well known. The results based on degree theory need the operator to be defined in a suitable set with nonempty interior, and this can be quite restrictive for certain applications. For example, when we are looking for positive solutions to boundary value problems in cones of L^p spaces, these cones could have empty interior and so degree techniques become inapplicable. If X is a retract of a Banach space E, the fixed point index enables one to mimic various degree theory properties, even though X may have an empty interior in E. For these reasons, it is interesting to consider the problem of generalizing this index to different classes of maps. Recently, in 2019, Djebali and Mebarki extended the fixed point index for the sum of an expansive operator and a k-set contraction (see [34]). Their results had a significant impact on the theory of fixed point on cones for the sum of two operators. This new index allows us to obtain new fixed point results.

This thesis is divided into three parts. The first part is devoted to the general framework, where we present some concepts and basic tools that will be used in the next parts. In Section 1.1, we give some essential mathematical tools, starting with the concept of cones, which is essential to introduce a partial ordering on abstract Banach spaces. In Section 1.2, we recall some compactness criteria, and after that, we introduce the three main and most commonly used measures of noncompactness and recall their basic properties. In Section 1.3, we define some classes of operators. In Section 1.4, we present the fixed point index theory, starting with retraction, and then define the fixed point index for strict set contractions on a retract of a Banach space. Then we present the fixed point index on cone for a k-set contraction perturbed by an operator T, where $I - T$ is Lipschitz invertible. Next, we give some computations of this fixed point index on the translate of a cone.

The second part of this thesis is devoted to difference equations, it contains three chapters: In chapter 2, we recall some mathematical background about linear difference equations. We start with some examples of the application of difference equations in the real world, then we give a survey of the fundamentals of the difference calculus: the difference operator, the summation operator and some methods for solving the linear difference equations. Then we present the theory of self adjoint difference equation on a discrete interval,

$$\Delta(p(t-1)\Delta y(t-1)) + q(t)y(t) = 0,\ t \in [a, b+1] = \{a, a+1, \ldots, b+1\}, \tag{1}$$

including some concepts such as generalized zero, disconjugacy, the Cauchy function and the calculation of the Green functions for some boundary value problems.

In chapter 3, we use a new topological approach, on fixed point index for the sum of two operators, to establish new existence criteria for the existence of positive solutions for the class of first order impulsive difference equations with a family of nonlinear boundary conditions:

$$\begin{aligned} \Delta x(n) &= f(n, x(n)), \quad n \neq n_k, \quad n \in J, \\ \Delta x(n_k) &= I_k(x(n_k)), \quad n = n_k, \\ Mx(0) - Nx(T) &= g(x(0), x(T)), \end{aligned} \tag{2}$$

where Δ is the forward difference operator, i.e., $\Delta u(n) = u(n+1) - u(n)$, $J = [0, T] \cap \mathbb{N}$, $T \in \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers, $M, N > 0$, $f \in \mathcal{C}(J \times \mathbb{R})$, $g \in \mathcal{C}(\mathbb{R} \times \mathbb{R})$, $I_k \in \mathcal{C}(\mathbb{R})$, $k \in \{1, \ldots, p\}$, $\{n_k\}_{k=1}^{p}$ are fixed impulsive points such that

$$0 < n_1 < n_2 < \ldots < n_p < T, \quad p \in \mathbb{N}.$$

In chapter 4, we first give an extension of Avery-Anderson theorem to the sum of two operators by making use of the fixed point index developed for operators that are sums of the form $T+S$, where $I-T$ is a Lipschitz invertible mapping and S is a k-set contraction. Then the obtained result is applied to prove the existence of at least one positive solution for the class of non-autonomous second order difference equations with Dirichlet boundary conditions:

$$\triangle^2 u(k) + f(k, u(k)) = 0, \quad k \in \{0, 1, \ldots, N\},\ N \in \mathbb{N}, N > 1. \tag{3}$$

$$u(0) = u(N+2) = 0. \tag{4}$$

where $f : \{0, ..., N+2\} \times [0, \infty) \to [0, \infty)$ is a continuous function.

The third part of this thesis is devoted to differential equations, it contains three chapters: In Chapter 5, we first develop new multiple fixed point theorems for the sum of two operators $T+S$, where $I-T$ is Lipschitz invertible and S is a k-set contraction on translate of a cone of a Banach space. Then the obtained results are applied to discuss the existence of multiple nontrivial positive solutions for the generalized Sturm-Liouville multipoint boundary value problem:

$$\begin{array}{rcl} -u''(t) & = & h(t)f(t, u(t), u'(t)), \quad 0 < t < 1, \\ au(0) - bu'(0) & = & \sum\limits_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) & = & \sum\limits_{i=1}^{m-2} b_i u(\xi_i), \end{array} \tag{5}$$

where $f \in \mathcal{C}([0,1] \times \mathbb{R}^* \times \mathbb{R}, \mathbb{R})$, $h \in \mathcal{C}([0,1], \mathbb{R})$, $a, b, c, d \in [0, \infty)$, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1\,(m \geq 3)$, $a_i, b_i \in [0, \infty)$ are constants for $i = 1, 2, \ldots, m-2$ and $\rho = ac + ad + bc > 0$.

In Chapter 6, we present a generalization of the functional expansion-compression fixed point theorem developed by Avery et al. in [11] to the case of a k-set contraction perturbed by an operator T, where $I-T$ is Lipschitz invertible, then fixed points are localized in sets of the form:

$$\mathcal{A}(\beta, b, \alpha, a) = \{x \in \mathcal{A} :\ a < \alpha(x) \text{ and } \beta(x) < b\},$$

with $\mathcal{A}$ is a relatively open subset of a cone $\mathcal{P}$, α and β are nonnegative continuous concave and convex functionals on $\mathcal{P}$, respectively. Next, we apply the obtained result to discuss the existence of at least one nontrivial positive solution to a non-autonomous second order boundary value problem:

$$x''(t) + f(t, x(t)) = 0, \quad t \in (0,1), \tag{6}$$

$$x(0) = x'(1) = 0, \tag{7}$$

where $f : (0,1) \times [0,\infty) \to [0,\infty)$ is a continuous function.

In Chapter 7, we investigate the following eigenvalue fourth order singular differential equation with boundary conditions at two points

$$v^{(4)}(t) = \lambda g(t) f(v(t)), \quad 0 < t < 1, \tag{8}$$

$$v(0) = a_1, \quad v(1) = a_2, \quad v''(0) = a_3, \quad v''(1) = a_4, \tag{9}$$

where $f \in \mathcal{C}([0,1])$, $g : (0,1) \to \mathbb{R}^+$ is continuous and may be singular at $t = 0$ or/and $t = 1$ and $a_j \geq 0$, $j \in \{1,2,3,4\}$ are given constants. Our existence result of at least one positive solution is based on a recent Birkhoff-Kellogg type fixed point theorem developed on translate of a cone on a Banach space.

Part I

General concepts

1 Essential Mathematics tools

1.1 Cones and partial ordering

Cones are closed convex subsets that can be used to generate a partial order in a linear normed space. Usually, this concept is used in searching for positive solutions of nonlinear differential and difference equations (see [4, 59]). In all what follows we denotes by $(E, \|.\|)$ a Banach space.

Definition 1.1. *A nonempty subset $\mathcal{P} \subset E$ is called a cone if it is closed, convex and satisfies the two following conditions:*

(i) $(x \in \mathcal{P}$ *and* $\lambda \geq 0) \Rightarrow \lambda x \in \mathcal{P}$,

(ii) $(x \in \mathcal{P}$ *and* $-x \in \mathcal{P}) \Rightarrow x = 0$, *i.e.*, $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$.

Definition 1.2. *Given a cone $\mathcal{P}$ of E, we define a partial ordering $\leq$ with respect to $\mathcal{P}$ in the following manner:*

$$\forall x, y \in E : \ x \leq y \Leftrightarrow y - x \in \mathcal{P}.$$

We can also define the following order relations:

- $x < y \Leftrightarrow x \leq y$ *and* $x \neq y$.
- $x \ll y \Leftrightarrow y - x \in \mathring{\mathcal{P}}$ *if* $\mathring{\mathcal{P}} \neq \emptyset$.
- $x \nleq y \Leftrightarrow y - x \notin \mathcal{P}$.

A segment of a cone $\mathcal{P}$ (order interval) is given by:

$$[x, y] = \{z \in \mathcal{P} : \ x \leq z \leq y\}.$$

Definition 1.3. *Let $\mathcal{P}$ be a cone of E. We say that:*

- *$\mathcal{P}$ is normal if there existe a constant $\delta > 0$ such that*

  $$\|x + y\| \geq \delta, \ \forall x, y \in \mathcal{P}, \ \textit{with} \ \|x\| = \|y\| = 1.$$

  Geometrically, normality means that the angle between two positive unit vectors has to be bounded away from π. In other words, a normal cone cannot be too large.

- *$\mathcal{P}$ is solid if its interior is nonempty, i.e., $\mathring{\mathcal{P}} \neq \emptyset$.*

- *$\mathcal{P}$ is generating if $E = \mathcal{P} - \mathcal{P}$, i.e., every element $x \in E$ can be represented in the form $x = u - v$, where $u, v \in \mathcal{P}$.*

Theorem 1.1. *([48] Theorem 1.1.1, p. 2) Let $\mathcal{P}$ be a cone in E. Then the following properties are equivalent:*

1. *$\mathcal{P}$ is normal.*

2. *There exists $\gamma > 0$ such that $\|x + y\| \geq \gamma \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{P}$.*

3. *There exists a constant $N > 0$ such that*

   $$0 \leq x \leq y \Longrightarrow \|x\| \leq N\|y\|, \ \textit{for all} \ x, y \in \mathcal{P}, \tag{1.1}$$

   i.e., the norm $\|.\|$ is semi monotone.

4. *There exists an equivalent norm $\|.\|_1$ on E such that $0 \leq x \leq y \Longrightarrow \|x\|_1 \leq \|y\|_1$, i.e., the norm $\|.\|_1$ is monotone.*

5. *$x_n \leq z_n \leq y_n \ (n = 1, 2, 3, ...)$, $\|x_n - x\| \to 0, \|y_n - x\| \to 0 \Longrightarrow \|z_n - x\| \to 0$.*

6. *A set $(B + \mathcal{P}) \bigcap (B - \mathcal{P})$ is bounded, where $B = \{x \in E : \ \|x\| \leq 1\}$.*

7. *Any order interval $[x, y] = \{z \in E : \ x \leq z \leq y\}$ is bounded.*

Remark 1.1. *The least positive number N satisfying (1.1) is called a normal constant. Clearly, $N \geq 1$. In fact, taking $y = x \neq 0$ in (1.1), we have $N \geq 1$.*

Example 1.1. 1. *Let $E = \mathbb{R}^n$ and $\mathcal{P}_1 = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n : \ x_i \geq 0, \ i = 1, ..., n\} = (\mathbb{R}_+)^n$.*

(a) *$\mathcal{P}_1$ is a solid and a generating cone in $\mathbb{R}^n$, since $\mathring{\mathcal{P}}_1 = (\mathbb{R}_+^*)^n$ and $\mathbb{R}_+$ is a generating cone on $\mathbb{R}$, then for $i = 1, ..., n : \forall x_i \in \mathbb{R}, \exists u_i, v_i \in \mathbb{R}_+ : x_i = u_i - v_i$.*

(b) *Moreover, all the norms on $\mathbb{R}^n$ are monotonic, we have that*

$$\forall x, y \in \mathbb{R}^n, \, 0_{\mathbb{R}^n} \leq x \leq y \Rightarrow \|x\| \leq \|y\|.$$

Hence $\mathcal{P}_1$ is normal and the normal constant $N = 1$.

2. *Let $E = \mathcal{C}(G)$, be the set of continuous functions on a closed and bounded subset $G \subset \mathbb{R}^n$, endowed with the norm $\|x\|_{\mathcal{C}(G)} = \sup\limits_{t \in G} |x(t)|$ et $\mathcal{P}_2 = \{x \in \mathcal{C}(G) : x(t) \geq 0, \ \forall t \in G\}$.*

(a) *$\mathcal{P}_2$ is a solid and a generator cone on $\mathcal{C}(G)$.*

(b) *$\mathcal{P}_2$ is normal because its norm $\|.\|_{\mathcal{C}(G)}$ is monotonic on $\mathcal{C}(G)$.*

(c) *We define additional cones on E such that:*

$$\mathcal{P}_3 = \{x \in \mathcal{C}(G) : x(t) \geq 0, \ \text{and} \int_{G_0} x(t)dt \geq \varepsilon_0 \|x(t)\|_{\mathcal{C}(G)}\},$$

$$\mathcal{P}_4 = \{x \in \mathcal{C}(G) : x(t) \geq 0, \ \text{and} \ \min_{t \in G_1} x(t) \geq \varepsilon_1 \|x(t)\|_{\mathcal{C}(G)}\}.$$

with $G_0, G_1$are closed subsets of G, and ε_0 and ε_1 are two constants such that $0 < \varepsilon_0 < mes(G_0)$ and $0 < \varepsilon_1 < 1$. We have $\mathcal{P}_3 \subset \mathcal{P}_2$ and $\mathcal{P}_4 \subset \mathcal{P}_2$ are both normal solid cones on $\mathcal{C}(G)$.

3. *Let $E = L^p(\Omega)$, be the set of Lebesgue integrable functions on $\Omega \subset \mathbb{R}^n$ with $p \geq 1$ and $0 < mes(\Omega) < \infty$ endowed with the norm $\|x\| = \left(\int_\Omega |x(t)|^p \, dt\right)^{\frac{1}{p}}$ and*

$$\mathcal{P}_5 = L_p^+(\Omega) = \{x \in L^p(\Omega) : \ x(t) \geq 0 \quad a.e. \ \text{in} \ \Omega\}.$$

It is clear that $\mathcal{P}_5$ is generator, and the norm of $L^p(\Omega)$ is increasing, then it is normal, but it is not a solid cone, because $\mathring{\mathcal{P}}_5 = \emptyset$ only the cone $L_\infty^+(\Omega)$ which has non empty interior.

In fact, if $\mathring{\mathcal{P}}_5$ *was of non empty interior then* $\exists f \in \mathring{\mathcal{P}}_5$*, i.e.* $\exists \delta > 0$ *such that* $\mathcal{B}(f, \delta) \subset \mathcal{P}_5$*. We take* $\Omega = [0, 1]$*, and consider the following sequence* $(f_n)_{n\in\mathbb{N}}$ *defined by:*

$$f_n(t) = \begin{cases} -f(t), & \text{if } t \in [0, \frac{1}{n}], \\ f(t), & \text{if } t \in]\frac{1}{n}, 1]. \end{cases}$$

Then

$$\begin{aligned} \int_0^1 |f_n(t) - f(t)|^p dt &= \int_0^{\frac{1}{n}} |f_n(t) - f(t)|^p dt + \int_{\frac{1}{n}}^1 |f_n(t) - f(t)|^p dt \\ &= \int_0^{\frac{1}{n}} |-f(t) - f(t)|^p dt + \int_{\frac{1}{n}}^1 |f(t) - f(t)|^p dt \\ &= 2^p \int_0^{\frac{1}{n}} |f(t)|^p dt. \end{aligned}$$

Hence $\|f_n - f\| = 2 \left(\int\limits_0^{\frac{1}{n}} |f(t)|^p dt \right)^{\frac{1}{p}} \to 0$ *when* $n \to \infty$*, if* $f \in L_p^+(\Omega)$*, that is*

$$\forall \delta > 0, \exists n_0 \in \mathbb{N}, n \geq n_0 \Rightarrow \|f_n - f\| \leq \delta,$$

which implies that

$$\forall \delta > 0, \exists n_0 \in \mathbb{N}, n \geq n_0 \Rightarrow f_n \in \mathcal{B}(f, \delta),$$

which contradicts the fact that f_n *is not in the cone* $\mathcal{P}_5$*, because mes* $\left([0, \frac{1}{n}]\right) \neq 0$*.*

4. *Let* $E = \mathcal{C}^1[0, 2\pi]$*, be the space of all continuous and differentiable functions defined on* $[0, 2\pi]$*, where its norm is defined as follows:*

$$\|x\| = \max_{0 \leq t \leq 2\pi} |x(t)| + \max_{0 \leq t \leq 2\pi} |x'(t)|.$$

Let $\mathcal{P}_6 = \{x \in \mathcal{C}^1[0, 2\pi] : x(t) \geq 0,\ 0 \leq t \leq 2\pi\}$*. Clearly* $\mathcal{P}_6$ *is a solid cone in* $\mathcal{C}^1[0, 2\pi]$*. $\mathcal{P}_6$ is not normal. In fact, if* $\mathcal{P}_6$ *is normal, then there exists a constant* $N > 0$ *such that, if* $0 \leq x \leq y$*, then* $\|x\| \leq N\|y\|$*. Let* $x_n(t) = 1 - \cos nt$ *and* $y_n(t) = 2$ *for* $n = 1, 2, \ldots$*. Clearly, we have*

$$0 \leq x_n \leq y_n,\ \|x_n\| = 2 + n,\ \|y_n\| = 2.$$

Then $2 + n \leq 2N$ *for* $n = 1, 2, \ldots$*, which is impossible. Therefore,* $\mathcal{P}_6$ *is not normal.*

1.2 Compactness and noncompactness

1.2.1 Some compactness criteria

Analysis is an immense field of mathematics, and compactness concepts and arguments enter in a great many different branches of analysis. Compactness also plays a vital role in many existence theorems as a technique of proof. Let us recall some well-known definitions and results.

Definition 1.4. *Let X be a topological space, $D \subset X$ any subset. A set U is relatively open in D if there is an open set Ω in X such that $U = \Omega \cap D$.*

Note that a set $U \subset D$ can be relatively open to D without being an open set of X. For example:

1. $D = [0, 1]$, the half-open interval $]a, 1]$ is relatively open in D for every $0 \leq a < 1$, since $]a, 1] = [0, 1] \cap]a, 3[$. It is clear that $]a, 1]$ is not open in $\mathbb{R}$.

2. $X = \mathbb{R}^2$, $D = \mathbb{R} \times \{0\}$ and $U =]0, 1[\times \{0\}$.

Definition 1.5. *A family of open sets $\{O_i : i \in I\}$ of a topological space (X, τ) is called an open cover for X, if $X = \bigcup\limits_{i \in I} O_i$. If each open cover of X has a finite subcover, then the topological space is called compact.*

Definition 1.6. *A locally compact space is a Hausdorff topological space with the property that each of its points admits a compact neighborhood.*

Definition 1.7.
- *A subset M of a metric space (X, d) is said to be totally bounded, if for each $\varepsilon > 0$, there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X (depending on ε) such that $M \subset \bigcup\limits_{i=1}^{n} B(x_i, \varepsilon)$. The set $\{x_1, x_2, ..., x_n\}$ is called a finite ε-net.*
- *A subset M is said to be bounded if $M \subset B(x, r)$ for some $x \in X$ and some $r > 0$.*

Theorem 1.2. *(Heine-Borel property) A subset M of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Definition 1.8. *A subset M of a topological space is said to be relatively compact if the closure $\overline{M}$ of M is a compact set.*

Proposition 1.1. *If the metric space (X, d) is complete, then the set M is relatively compact if and only if it is totally bounded.*

Theorem 1.3. *([73], Theorem 28.2, p.179) For a metric space (X, d), the following properties are equivalent:*

(i) *X is compact;*

(ii) *X is complete and totally bounded;*

(iii) *every sequence in X has a convergent subsequence;*

(iv) *X has the Bolzano-Weierstrass property, every infinite subset M of X has a limit point $x_0 \in X$, i.e. a point x_0 such that every neighbourhood of x_0 meets M.*

Definition 1.9. *Let X and Y be Banach spaces and M a subset of X. A mapping $T : M \subset X \to Y$ is called compact (or completely continuous) if T is continuous and maps bounded sets into relatively compact sets.*

Remark 1.2. *Compact operators are useful in nonlinear functional analysis. Many results about continuous operators on $\mathbb{R}^n$ are generalized to Banach spaces by replacing continuous operator with compact operator.*

Remark 1.3. *For finite dimensional Banach spaces, continuous and compact operators are the same whenever the domain of definition is closed. Indeed, if M is a bounded set, then $\overline{M}$ is compact. Thus $f(\overline{M})$ is also compact, and so $f(M)$ is relatively compact.*

We now apply this result to find the compact subspaces of the space $\mathcal{C}(X, \mathbb{R}^n)$, in the uniform topology. We know that a subspace of $\mathbb{R}$ is compact if and only if it is closed and bounded. One might hope that an analogous result holds for $\mathcal{C}(X, \mathbb{R}^n)$. But it does not, even if X is compact. One needs to assume that the subspace of $\mathcal{C}(X, \mathbb{R}^n)$ satisfies an additional condition, called equicontinuity which is introduced independently by Arzelà and Ascoli. We consider that notion now.

Definition 1.10. *The sequence* $(f_n)_{n\in\mathbb{N}} \subset \mathcal{C}([a,b],\mathbb{R})$ *is equicontinuous at* $x \in [a,b]$ *if for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon$ *for every* $n \in \mathbb{N}$*, and* $(f_n)_{n\in\mathbb{N}}$ *is called equicontinuous if it is equicontinuous at every* x *in* $[a,b]$.

Definition 1.11. *The sequence* $(f_n)_{n\in\mathbb{N}}$ *is uniformly bounded if there is an* $L > 0$ *such that* $|f_n(x)| < L$ *for all* $x \in [a,b]$ *and all* $n \in \mathbb{N}$*.*

In its original form, the Ascoli theorem provides conditions for a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous real valued functions on a closed interval $[a,b]$ to have a uniformly convergent subsequence. Therefore, the Ascoli-Arzelà quest is to find sufficient and necessary conditions on subsets $\mathcal{M}$ of a space $\mathcal{C}(X,Y)$ of continuous functions between two topological spaces to have compact closure, that is to be relatively compact, for some appropriate analogue of the topology of uniform convergence. The literature is rich in results of that type. In the following, some variants of Ascoli-Arzelà Theorem.

Theorem 1.4. *Let* (X,d) *be a compact metric space,* $(Y,\|.\|)$ *a Banach space, and the space* $\mathcal{C}(X,Y)$ *is endowed with the sup norm* $\|f\| = \sup\limits_{x\in X} \|f(x)\|_Y$*. A subset* $\mathcal{M} \subset \mathcal{C}(X,Y)$ *is relatively compact if and only if*

(a) $\mathcal{M}$ *is equicontinuous.*

(b) $\forall\, x \in X$*, the set* $\mathcal{M}(x) = \{f(x),\ f \in \mathcal{M}\}$ *is relatively compact in* Y*.*

Theorem 1.5. *Any bounded equicontinuous sequence of functions in* $\mathcal{C}([a,b],\mathbb{R})$ *has a uniformly convergent subsequence.*

Theorem 1.6. *A subset of* $\mathcal{C}([a,b],\mathbb{R})$ *is compact if and only if it is closed, bounded, and equicontinuous.*

Theorem 1.7. *([73], Theorem 47.1, p.290) Let* X *be a topological space and* Y *be a metric space. If* $\mathcal{M} \subset \mathcal{C}(X,Y)$ *is equicontinuous and pointwise bounded, then* $\mathcal{M}$ *is relatively compact in* $\mathcal{C}(X,Y)$*, that is in* $\mathcal{C}(X,Y)$ *endowed with the topology of uniform convergence on compact subsets of* X*. Moreover, if* X *is locally compact, the converse is true.*

Theorem 1.8. *Let* $\mathcal{M} \subset \mathcal{C}^1([a,b],\mathbb{R})$ *satisfy the following conditions:*

(a) *There exists $L > 0$ such that for all $t \in [a,b]$ and $u \in \mathcal{M}$, $|u(t)| \leq L$ and $|u'(t)| \leq L$.*

(b) *For every positive $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < \delta(\varepsilon)$ and for all $u \in \mathcal{M}$,*

$$|u(t_1) - u(t_2)| \leq \varepsilon \text{ and } |u'(t_1) - u'(t_2)| \leq \varepsilon.$$

Then the set $\mathcal{M}$ is relatively compact in $\mathcal{C}^1([a,b],\mathbb{R})$.

Theorem 1.9. *([29], p.62) Let $\mathcal{M} \subset \mathcal{C}_b(\mathbb{R}^+,\mathbb{R})$. Then $\mathcal{M}$ is relatively compact in $\mathcal{C}_b(\mathbb{R}^+,\mathbb{R})$ if the following conditions hold:*

(i) $\mathcal{M}$ is uniformly bounded in $\mathcal{C}_b(\mathbb{R}^+,\mathbb{R})$.

(ii) The functions belonging to $\mathcal{M}$ are almost equicontinuous on $\mathbb{R}^+$, i.e. equicontinuous on every compact interval of $\mathbb{R}^+$.

(iii) The functions from $\mathcal{M}$ are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - l| < \varepsilon$ for every $t \geq T(\varepsilon)$ and $x \in \mathcal{M}$.

In the following result, E is a Banach space (not necessarily finite dimensional), we define the discrete interval $[0, T+1] = \{0, 1, \ldots, T+1\}$.

Theorem 1.10. *([2], Theorem 17.1, p.262)(Discrete Ascoli-Arzelà Theorem)*
Let $\mathcal{M}$ be a closed subset of $\mathcal{C}([0,T+1],E)$. If $\mathcal{M}$ is uniformly bounded and the set $\{u(k) : u \in \mathcal{M}\}$ is relatively compact for each $k \in [0,T+1]$, then $\mathcal{M}$ is compact.

Proof. We need only to show that every sequence in $\mathcal{M}$ has a Cauchy subsequence. Let $M_1 = \{f_{1,1}, f_{1,2}, \ldots\}$ be any sequence in $\mathcal{M}$. Notice the sequence $\{f_{1,j}(0)\}$, $j = 1, 2, \ldots$ has a convergente subsequence and let $M_2 = \{f_{2,1}, f_{2,2}, \ldots\}$ denote this subsequence. For $\{f_{2,j}(1)\}$, $j = 1, 2, \ldots$ let $M_3 = \{f_{3,1}, f_{3,2}, \ldots\}$ be a subsequence of M_2 such that $\{f_{3,j}(1)\}$ converges. Since M_3 is a subsequence of M_2 then $\{f_{3,j}(0)\}$ also converges. Continue this process to get a list of subsequences

$$M_1, M_2, \ldots, M_{T+2}, M_{T+3}$$

in which each sequence is a subsequence of the one directly on the left of it and for each k, the sequence $M_k = \{f_{k,1}, f_{k,2}, \ldots\}$ has the property that $\{f_{k,j}(k-2)\}, \quad j = 1, 2, \ldots$ is a convergente sequence. Thus for each $k \in [0, T+1]$, the sequence $\{f_{T+3,j}(k)\}$ is convergent. Then since $\{f_{T+3,j}(k)\}$ is Cauchy for each $k \in [0, T+1]$, and since $[0, T+1]$ is finite, we have that there exists $n_0 \in \{1, 2, \ldots\}$ independent of k such that

$$m, n \geq n_0 \text{ implies } |f_{T+3,m}(k) - f_{T+3,n}(k)| < \varepsilon, \quad k \in [0, T+1].$$

Thus M_{T+3} is Cauchy. □

1.2.2 Measures of noncompactness

Let us recall that a subset A of a Banach space X is relatively compact if and only if to every $\varepsilon > 0$ there are finitely many balls of radius ε such that their union covers A. If A is only bounded, there is a positive lower bound for such numbers ε. These facts suggest to introduce a new concept called measure of noncompactness (MNC for short), which determines the deviation from relative compactness for a set.

Definition 1.12. *Let $\mathcal{A}$ be the family of bounded sets of a Banach space E. A function $\psi : \mathcal{A} \to [0, +\infty[$ is called measure of noncompactness if it satisfies the following conditions:*

1. $\psi(A) = 0 \Leftrightarrow A$ *is relatively compact,* $\forall A \in \mathcal{A}$.
2. $\psi(A) = \psi(\overline{A})$, $\forall A \in \mathcal{A}$.
3. $\psi(A_1 \cup A_2) = \max\{\psi(A_1), \psi(A_2)\}$, $\forall A_1, A_2 \in \mathcal{A}$.

In this section, we present the three main and most frequently used MNCs. The first measure of noncompactness, the function α, was defined and studied by Kuratowski [61] in 1930. In 1955, the Italian mathematician Darbo [30] used the Kuratowski measure to generalize Schauder's and Banach's fixed point theorems to strict set operators. The Hausdorff MNC γ was introduced by Goldenstein et al. [45] in 1957, and the Istratescu MNC χ introduced by Istratescu [54] in 1972.

In the sequel, we recall briefly the definitions of the above measures of noncompactness and their basic properties.

Definition 1.13. ***(Kuratowski MNC)*** *Let E be a real Banach space. The Kuratowski MNC of a nonempty and bounded subset A of E, denoted by $\alpha(A)$ is defined by*

$$\alpha(A) = \inf\{\varepsilon > 0 : \ A = \bigcup_{i=1}^{n} A_i, \ A_i \subset E \text{ and } diam(A_i) \leq \varepsilon, \text{for all } i = 1, ..., n, \ n \in \mathbb{N}\} \quad (1.2)$$

where $diam(A_i) = \sup\{\|x-y\|_E, \ x, y \in A_i\}$ is the diameter of A_i. i.e., the Kuratowski MNC of a nonempty and bounded subsets A of E, denoted by $\alpha(A)$, is the infimum of the all numbers $\varepsilon > 0$ such that A admits a finite covering by sets of diameter smaller than ε.

Example 1.2. *For the closed unit ball B, in infinite dimensional Banach space, we have*

$$\alpha(B) = \alpha(\partial B) = \alpha(\mathring{B}) = 2.$$

However, in a finite dimensional Banach space,

$$\alpha(B) = \alpha(\partial B) = \alpha(\mathring{B}) = 0.$$

Remark 1.4. *In general, the computation of the exact value of $\alpha(A)$ is difficult.*

Another measure of noncompactness, which seems to be more applicable, is so-called Hausdorff measure of noncompactness (or ball measure of noncompactness). It is defined as follows.

Definition 1.14. ***(Hausdorff MNC)*** *Let E be a Banach space. The Hausdorff MNC of a nonempty and bounded subset A of E, denoted by $\gamma(A)$, is the infimum of all reals $\varepsilon > 0$ such that A can be covered by a finite number of balls with radius $< \varepsilon$, that is,*

$$\gamma(A) = \inf\{\varepsilon > 0 : A \subset \bigcup_{i=1}^{n} B(x_i, r_i), \ x_i \in E, \ r_i < \varepsilon, i = 1, 2, ..., n, \ n \in \mathbb{N}\} \quad (1.3)$$

Remark 1.5.

- *In the definition of the Hausdorff MNC of the set A it is not supposed that centers of the balls that cover A belong to A. Hence (1.3) is equivalent to*

  $$\gamma(A) = \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{-net in } E\},$$

  where by ε-net, we mean a set $\{a_1, a_2, \dots, a_n\} \subset E$ such that the balls $B(a_1, \varepsilon); B(a_2, \varepsilon); \dots ; B(a_n, \varepsilon)$ cover A.

- *In the definition of the Hausdorff MNC, instead of a finite ε-net one can speak of a totally bounded one, i.e., an ε-net A that has a finite δ-net for any $\delta > 0$.*

Definition 1.15. ***(Inner Hausdorff MNC)*** *Let E be a Banach space. The Inner Hausdorff MNC of a nonempty and bounded subset A of E, denoted by $\gamma_0(A)$, is the infimum of all reals $\varepsilon > 0$ such that A can be covered by a finite number of balls with radius $< \varepsilon$ and centers in A, that is,*

$$\gamma_0(A) = \inf\{\varepsilon > 0 : A \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in A,\ r_i < \varepsilon, i = 1, 2, ..., n,\ n \in \mathbb{N}\} \tag{1.4}$$

Remark 1.6. *The formula in (1.4) is equivalent to*

$$\gamma_0(A) = \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{-net in } A\}.$$

Definition 1.16. ***(Istratescu MNC)*** *Let E be a Banach space, the Istratescu MNC also called lattice MNC of a nonempty and bounded subset A of E, denoted by $\chi(A)$, is defined by*

$$\chi(A) = \sup\{\rho > 0 : \text{ there exists a sequence } (x_n)_n \in A \text{ such that } \|x_m - x_n\| \geq \rho \text{ for } m \neq n\}$$

We list below some of the properties of the MNCs α and γ that follow immediately from the definition. These properties are straightforward consequences of others (for example, non-singularity follows from regularity, monotonicity from semi-additivity, continuity from Lipschitzianity), its proof can be found in [44].

Proposition 1.2. *Let $\mathcal{A}$ be the family of bounded sets of a real Banach space E, let ψ denote the MNC α or the MNC γ.*

(a) (Regularity). $\psi(A) = 0 \Leftrightarrow \bar{A}$ *is compact.*

(b) (Non-singularity). *If A is a finite set, then $\psi(A) = 0$.*

(c) (Semi-additivity). $\psi(A_1 \cup A_2) = \max\{\psi(A_1), \psi(A_2)\}$.

(d) (Monotonicity). $A_1 \subset A_2 \Rightarrow \psi(A_1) \leq \psi(A_2)$.

(e) (Lischitzianity). *Let A_1 and A_2 be subsets of $\mathcal{A}$, then $|\psi(A_1) - \psi(A_2)| \leq L_\psi d_H(A_1, A_2)$, where $L_\alpha = 2, L_\gamma = 1$ and d_H denotes the Hausdorff metric defined by*

$$d_H(A_1, A_2) = \max\{\sup_{x\in A_1} d(x, A_2), \sup_{y\in A_2} d(y, A_1)\},$$

where $d(.,.)$ denotes the distance from an element of E to a subset of E.i.e., $d(x, A_2) = \inf_{y\in A_2} d(x, y)$ and $d(y, A_1) = \inf_{x\in A_1} d(y, x)$.

(f) (Continuity). *For any $A \subset E$ and any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\psi(A)-\psi(A_1)| < \epsilon$ for all A_1 satisfying $d_H(A, A_1) < \delta$.*

(g) (Semi-homogeneity). *$\psi(\lambda A) = |\lambda|\psi(A)$ for any $\lambda \in \mathbb{R}$.*

(h) (Algebraic semi-additivity). *$\psi(A_1 + A_2) \leq \psi(A_1) + \psi(A_2)$.*

(i) (Invariance under translations). *$\psi(A + x) = \psi(A)$ for any $x \in E$.*

(j) (Invariance under passage to the closure). *$\psi(A) = \psi(\bar{A})$.*

(k) (Invariance under passage to the convex hull). *$\psi(A) = \psi(conv(A))$.*

To get an idea of how to calculate these measures of noncompactness, we give some examples.

Example 1.3. *Let $X = \mathcal{C}[0, 1]$ be the Banach space of all continuous real functions on $[0, 1]$, endowed with the maximum norm. For $M = B(X) = \{x \in X : \|x\| \leq 1\}$, we have*

$$\gamma(M) = \gamma_0(M) = 1, \ \alpha(M) = 2.$$

On the other hand, the set $M_1 = \{u \in B(X) : 0 = u(0) \leq u(t) \leq u(1) = 1\}$ satisfies

$$\gamma(M_1) = \frac{1}{2}, \ \gamma_0(M_1) = \alpha(M_1) = 1.$$

Similarly, for the set $M_2 = \{u \in B(X) : 0 \leq u(0) \leq \frac{1}{3}, \ 0 \leq u(t) \leq 1, \ \frac{2}{3} \leq u(1) \leq 1\}$ we obtain

$$\gamma(M_2) = \frac{1}{2}, \ \gamma_0(M_2) = \frac{2}{3}, \ \alpha(M_2) = 1.$$

Finally, the (noncompact) set

$$M_3 = \{u \in B(X) : \ 0 \leq u(t) \leq \frac{1}{2} \text{ for } 0 \leq t \leq \frac{1}{2}, \text{ and } \frac{1}{2} \leq u(t) \leq 1 \text{ for } \frac{1}{2} \leq t \leq 1\}$$

satisfies

$$\gamma(M_3) = \gamma_0(M_3) = \alpha(M_3) = \frac{1}{2}.$$

1.3 On some classes of operators

Definition 1.17. *Let E and F be Banach spaces and $\mathcal{A}$ be the family of bounded sets of E. Let $f : E \to F$ be a continuous and bounded mapping*

(a) f is called a k-set contraction, for some number $k \geq 0$, if $\alpha(f(A)) \leq k\alpha(A),\ \forall\, A \in \mathcal{A}$.

(b) f is called a 1-set contraction, if $k = 1$.

(c) f is called a strict set contraction if $0 \leq k < 1$.

(d) f is called a condensing, if $\forall\, A \in \mathcal{A}$ with $\alpha(A) > 0$, we have $\alpha(f(A)) < \alpha(A)$.

Some Lipschitz Invertible Mappings

Let X be a linear normed space and I be the identity map of X: The case of expansive mapping is given in the following lemma.

Definition 1.18. *Let (X, d) be a metric space and D be a subset of X. The mapping $T : D \to X$ is said to be expansive if there exists a constant $h > 1$ such that*

$$d(Tx, Ty) \geq h\, d(x, y), \quad \forall\, x, y \in D.$$

Example 1.4.

(1) An affine function with a leading coefficient $\beta > 1$ is β-expansive on $\mathbb{R}$.

(2) The function $f(x) = x^3 + \sigma x,\ x \in \mathbb{R}^+$ is σ-expansive for $\sigma > 1$.

(3) The function $f(x) = \gamma\, \frac{x}{x+\delta},\ x \in [a, b]$ is $\frac{|\gamma\, \delta|}{(b+\delta)^2}$-expansive.

Definition 1.19. *Let $(X, \|.\|)$ be a linear normed space and $D \subset X$. An operator $T : D \to X$ is said to be γ-Lipschitz invertible on D if it is invertible and its inverse is Lipschitzian on $T(D)$ with constant γ.*

In what follows we give some examples.

Example 1.5.

(1) The function $f(x) = \tan(x),\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is 1-Lipschitz invertible on $\mathbb{R}$.

(2) An affine function with a leading coefficient β is $\frac{1}{\beta}$-Lipschitz invertible on $\mathbb{R}$.

Lemma 1.1. *[89, Lemma 2.1] Let $(X, \|.\|)$ be a normed linear space, $D \subset X$. If a mapping $T : D \to X$ is expansive with a constant $h > 1$, then the mapping $I - T : D \to (I - T)(D)$ is invertible and*

$$\|(I-T)^{-1}x - (I-T)^{-1}y\| \leq (h-1)^{-1}\|x-y\| \ \textit{for all} \ \ x, y \in (I-T)(D).$$

Proof. For each x, $y \in D$, we have

$$\begin{aligned} \|(I-T)x - (I-T)y\| &= \|(Tx - Ty) - (x-y)\| \\ &\geq (h-1)\|x-y\|, \end{aligned} \tag{1.5}$$

which shows that $(I-T)^{-1} : (I-T)(D) \to D$ exists. Hence, for $x, y \in (I-T)(D)$, we have $(I-T)^{-1}x$, $(I-T)^{-1}y \in D$. Thus, using $(I-T)^{-1}x$, $(I-T)^{-1}y$ substitute for x, y in (1.5), respectively, we obtain

$$\|(I-T)^{-1}x - (I-T)^{-1}y\| \leq \frac{1}{h-1}\|x-y\|.$$

□

Some other examples of Lipschitz invertible mappings (see [88]) are presented below.

1. Let $(E, \|.\|)$ be a Banach space and $T : E \to E$ be Lipschitzian map with constant $\beta > 0$. Assume that for each $z \in E$, the map $T_z : E \to E$ defined by $T_z x = Tx + z$ satisfies that T_z^p is expansive for some $p \in \mathbb{N}$ and is surjective. Then $(I - T)$ maps E onto E, the inverse of $I - T : E \to E$ exists, and

$$\|(I-T)^{-1}x - (I-T)^{-1}y\| \leq \gamma_p\|x-y\| \ \text{for all} \ \ x, y \in E,$$

where

$$\gamma_p = \frac{\beta^p - 1}{(\beta - 1)(lip(T^p) - 1)},$$

where $lip(T^p) = \max\{h \geq 0 : \ d(T^p x, T^p y) \geq h\, d(x, y), \ \forall x, y \in E\}$.

2. Let $(X, \|.\|)$ be a linear normed space, $M \subset X$. Assume that the mapping $T : M \to X$ is contractive with a constant $k < 1$, then the inverse of $I - T : M \to (I-T)(M)$ exist, and $\|(I-T)^{-1}x - (I-T)^{-1}y\| \leq (1-k)^{-1}\|x-y\|$ for all $x, y \in (I-T)(M)$.

3. Let $(E, \|.\|)$ be a Banach space and $T : E \to E$ be Lipschitzian map with constant $\beta \geq 0$. Assume that for each $z \in E$, the map $T_z : E \to E$ defined by $T_z x = Tx + z$ satisfies that T_z^p is contractive for some $p \in \mathbb{N}$. Then $(I - T)$ maps E onto E, the inverse of $I - T : E \to E$ exists, and $\|(I-T)^{-1}x - (I-T)^{-1}y\| \leq \rho_p \|x - y\|$ for all $x, y \in E$, where

$$\rho_p = \begin{cases} \frac{p}{1-Lip(T^p)}, & \text{if } \beta = 1; \\ \frac{1}{1-\beta}, & \text{if } \beta < 1; \\ \frac{\beta^p - 1}{(\beta-1)(1-Lip(T^p))}, & \text{if } \beta > 1. \end{cases}$$

where $Lip(T^p)$ denotes the Lipschitz constant for T^p if T^p is a Lipschitz map.

1.4 Fixed point index theory

The fixed point index theory has proved to be a useful tool in studying nonlinear problems. It is so important for obtaining existence theorems for solutions of equations, that many mathematicians have labored to extend this concept to the most general situations conceivable for instance: strict set contraction, condensing mapping, 1-set contraction, k-set contraction and even for the sum of two operators [34, 43, 16].

1.4.1 On retract sets

Karel Borsuk introduced in 1931 the following notion, which is a kind of nonlinear projection.

Definition 1.20. *Let X be a Banach space and $r : X \to M$ is a continuous map with $M \subset X$. The map r is called a retraction if and only if*

$$r(x) = x, \quad \forall x \in M.$$

In that case, the set M is called a retract of X.

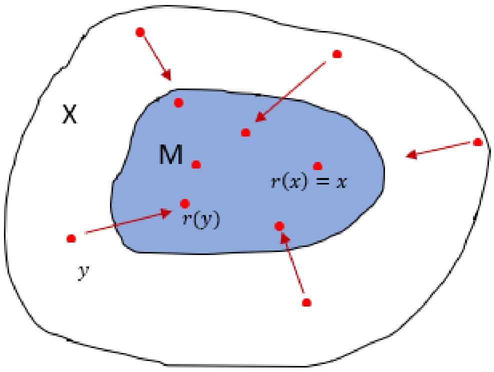

Figure 1.1

Example 1.6. *Let $X = \mathbb{R}^n$, $R > 0$, and $M = \{x \in X : \|x\| \leq R\}$. Then M is a retract of X. A retraction $r : X \to M$ is given by*

$$r(x) = \begin{cases} x, & \text{if } \|x\| \leq R, \\ \frac{Rx}{\|x\|}, & \text{if } \|x\| > R. \end{cases}$$

Example 1.7. *A retract of the punctured plane $\mathbb{R}^2 \backslash \{(0,0)\}$ to the circle $S^1 = \{(x,y),\ x^2+y^2 = 1\}$ is given by the retraction*

$$r(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

Example 1.8. *The projection of a square on its side AB is the mapping taking each point x of the square into the endpoint $f(x)$ of the perpendicular dropped from x on AB (see the Figure 1.2). For any ε-neighborhood of the point $f(x)$ the δ-neighborhood of the point x of the same radius projects into the first neighborhood. Hence, for any $\varepsilon > 0$ there exists a corresponding $\delta > 0$, in this case equal to ε. Therefore the projection map is continuous at each point of the square. Hence, a closed interval is a retract of a square.*

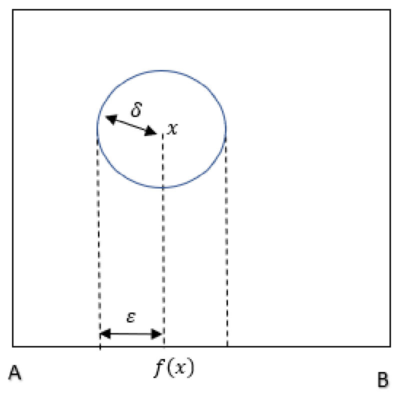

Figure 1.2

Remark 1.7. 1. *A closed interval is a retract of a triangle, as well as a convex polygon, a cube, etc. A circle is a retract of an annulus, (i.e., the region bounded by a pair of concentric circles).*

2. *The boundary of a disc is not a retract of a disc (see [81, Example 3]).*

Definition 1.21. *A space X has the fixed point property (or is a fixed point space) if every continuous map $f : X \to X$ has a fixed point.*

Remark 1.8. 1. *If Y has the fixed point property and X is a retract of Y, then X has the fixed point property. In fact, since X is a retract of Y, there is a continuous map $r : Y \to X$ so that $r = I$ on X. Let $f : X \to X$ be any continuous map. Then $f \circ r : Y \to X \subset Y$. Hence, using that Y has the fixed point property, there is $y \in Y$ so that $(f \circ r)(y) = y$ and $y \in X$. From here, $r(y) = y$ and $(f \circ r)(y) = f(r(y)) = f(y) = y$. Thus f has a fixed point in X.*

2. *The significance of retractions lies in the ability to reduce fixed point questions for complicated sets to fixed point questions for simpler subsets [91, cf. proof of Prop.2.6 (II)].*

3. *Every closed convex subset of Banach space is a retract of that space (this is an easy consequence of Dugundji's extension theorem (see Theorem 1.11)). In particular, every cone or translate of cone in E is a retract of E.*

4. *Every retract is closed but not necessarily convex.*

Theorem 1.11. *(Dugundji's extension theorem)[31, Theorem 7.2, p. 44]Let X and Y be normed linear spaces, $A \subset X$ a closed subset and $f : A \to Y$ a continuous map. Then f has a continuous extension $\tilde{f} : X \to Y$ such that $\tilde{f}(X) \subset Conv(f(A))$.*

1.4.2 On the fixed point index

If X is a topological space, U is an open subset of X, and $f : U \to X$ is a continuous map which has a compact (possibly empty) fixed point set $S = \{x \in U : \ f(x) = x\}$, then one can define an integer $i(f, U, X)$, called the fixed point index of f in U with respect to X. Roughly speaking, $i(f, U, X)$ will be an algebraic count of the number of fixed points of f in U.

- If $U = X$, $i(f, X, X)$ is the famous Lefschetz number of $f : X \to X$.
- If X is a Banach space or normed linear space and $\overline{f(U)}$ is compact, then $i(f, U, X)$ is related to the Leray-Schauder degree of $I - f$, i.e. $i(f, U, X) = deg(I - f, U, 0)$ the Leray-Schauder degree of $I - f$ on U with respect to 0. Thus the fixed point index generalizes the Leray-Schauder degree. However, it is also defined in situations where degree theory is not directly applicable, for example, when X is a closed convex set with empty interior in a Banach space Y or, when X is not a vector subspace of Y.

Note that the fixed point index is a powerful tool for the study of nonlinear problems in analysis in comparison to Leray-Schauder degree. Many extend this index to larger classes like the class of strict-set contractions and that of 1-set contraction mappings.

1.4.3 Fixed point index for a k-set contraction perturbed by an operator T, where $I - T$ is Lipschitz invertible

In chapters 3, 4, 5 and 6, we will use the fixed point index for a k-set contraction perturbed by an operator T, where $I - T$ is Lipschitz invertible, either to develop new fixed point theorems, ensuring the existence of positive fixed points, or to discuss the existence of positive solutions of certain BVPs associated to differential or difference equations.

The purpose of this section is to present the definition of this index as well as some of its properties. The proofs of the results which will be presented involve the fixed point index for strict set contractions whose basic properties are collected in the following lemma. For the proof we refer the reader to [47, Theorem 1.3.5] or [3], [31], [48].

Lemma 1.2. *Let X be a retract of a Banach space E. For every bounded open subset $U \subset X$ and every strict set contraction $f : \overline{U} \to X$ without fixed points on ∂U, there exists uniquely one integer $i(f, U, X)$ satisfying the following conditions:*

(a) (Normalization). *For every constant map* $f : \overline{U} \to U$*, we have* $i\,(f, U, X) = 1$.

(b) (Homotopy invariance). *The index* $i(h(t, .), U, X)$ *does not depend on the parameter* $t \in [0, 1]$*, where*

(*i*) $h : [0, 1] \times \overline{U} \to X$ *is continuous and* $h(t, x)$ *is uniformly continuous in* t *with respect to* $x \in \overline{U}$,

(*ii*) $h(t, .) : \overline{U} \to X$ *is a strict* k*-set contraction, where* k *does not depend on* $t \in [0, 1]$,

(*iii*) $h(t, x) \neq x$, *for every* $t \in [0, 1]$ *and* $x \in \partial U$.

(c) (Additivity). *For every pair of disjoint open subsets* U_1, U_2 *in* U *such that* f *has no fixed points on* $\overline{U} \setminus (U_1 \cup U_2)$ *we have*

$$i\,(f, U, X) = i\,(f, U_1, X) + i\,(f, U_2, X),$$

where $i(f, U_k, X) := i(f_{|\overline{U}_k}, U_k, X)$, $k = 1, 2$.

(d) (Solvability). *If* $i\,(f, U, X) \neq 0$*, then* f *has at least one fixed point in* U.

(e) (Permanence). *If* Y *is a retract of* X *and* $f(\overline{U}) \subset Y$*, then*

$$i\,(f, U, X) = i\,(f, U \cap Y, Y),$$

$i\,(f, U \cap Y, Y) = i\,(f|_{\overline{U \cap Y}}, U \cap Y, Y)$.

(f) (Excision). *For every open subset* $V \subset U$ *such that* f *has no fixed point in* $\overline{U} \backslash V$*, then*

$$i\,(f, U, X) = i\,(f|_V, V, X).$$

Given a real Banach space $(E, \|.\|)$, let $\mathcal{P} \neq \{0\}$ be a cone in E and $\mathcal{K} = \mathcal{P} + \theta$ $(\theta \in E)$ a θ-translate of $\mathcal{P}$. The following results are direct consequences of the properties of the index i in case of a translate of a cone, rather than in a cone.

Proposition 1.3. *Let* $U \subset \mathcal{K}$ *be a bounded open subset with* $\theta \in U$*. Assume that* $A : \overline{U} \to \mathcal{K}$ *is a strict set contraction that satisfies the so-called Leray-Schauder boundary condition type:*

$$Ax - \theta \neq \lambda(x - \theta), \quad \forall\, x \in \partial U,\ \forall\, \lambda \geq 1. \tag{1.6}$$

Then $i\,(A, U, \mathcal{K}) = 1$.

Proof. Let

$$H(t,x) = tAx + (1-t)\theta, \quad t \in [0,1], \quad x \in \overline{U}.$$

We have $H : [0,1] \times \overline{U} \to \mathcal{K}$ is continuous and $H(t,\cdot) : \overline{U} \to \mathcal{K}$ is a strict set contraction. Assume that there is a $(t_0, x_0) \in [0,1] \times \partial U$ such that $H(t_0, x_0) = x_0$. Hence,

$$t_0 A x_0 + (1-t_0)\theta = x_0.$$

If $t_0 = 0$, then $x_0 = \theta$. This is a contradiction because $\theta \in U$.

If $t_0 \neq 0$, then $Ax_0 - \theta = \frac{1}{t_0}(x_0 - \theta)$, which contradicts with (1.6). Therefore $H(t,x) \neq x$ for any $(t,x) \in [0,1] \times \partial U$. Thus, by the homotopy invariance and the normality of the fixed point index, it follows

$$\begin{aligned} i(A,U,\mathcal{K}) &= i(H(1,.),U,\mathcal{K}) \\ &= i(H(0,.),U,\mathcal{K}) \\ &= i(\theta,U,\mathcal{K}) \\ &= 1. \end{aligned}$$

This completes the proof. □

Proposition 1.4. *Let $U \subset \mathcal{K}$ be a bounded open subset with $\theta \in U$. Assume that $A : \overline{U} \to \mathcal{K}$ is a strict set contraction that satisfies the condition of type norm:*

$$\|Ax - \theta\| \leq \|x - \theta\| \text{ and } Ax \neq x, \ \forall x \in \partial U. \tag{1.7}$$

Then $i(A,U,\mathcal{K}) = 1$.

Proof. It is sufficient to prove that the condition (1.7) implies the condition (1.6). Indeed, assume by contradiction that some $x_0 \in \partial U$ and $\lambda_0 \geq 1$ exist and satisfy $Ax - \theta = \lambda_0(x - \theta)$. We consider two cases: If $\lambda_0 = 1$, then $Ax_0 = x_0$, contradicting the hypothesis (1.7).

If $\lambda > 1$, then $\|Ax - \theta\| = \lambda_0 \|x - \theta\| > \|x_0 - \theta\|$, a contradiction of (1.7) is again reched. □

Proposition 1.5. *Let U be a bounded open subset of $\mathcal{K}$. Assume that $A : \overline{U} \to \mathcal{K}$ is a strict set contraction and there is $v_0 \in \mathcal{P}^*$ such that*

$$x - Ax \neq \lambda v_0, \ \textit{for all } x \in \partial U, \lambda \geq 0. \tag{1.8}$$

Then $i(A, U, \mathcal{K}) = 0$.

Proof. Define the homotopy $H : [0,1] \times \overline{U} \to \mathcal{K}$ by

$$H(t,x) = Ax + t\lambda_0 v_0,$$

for some

$$\lambda_0 > \|v_0\|^{-1} \sup_{x \in \overline{U}}((\|x\| + \|Ax\|)). \tag{1.9}$$

Such a choice is possible since U is a bounded subset and then so is $A(\overline{U})$. The operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t,.)$ is strict set contraction for each $t \in [0,1]$. In addition, $H(t,.)$ has no fixed point on ∂U. On the contrary, there would exist some $x_0 \in \partial U$ and $t_0 \in [0,1]$ such that

$$x_0 = Ax_0 + t_0 \lambda_0 v_0,$$

contradicting the hypothesis. By the homotopy invariance of the fixed point index, we have

$$\begin{aligned} i(A, U, \mathcal{K}) &= i(H(0,\cdot), U, \mathcal{K}) \\ &= i(H(1,\cdot), U, \mathcal{K}) \\ &= 0. \end{aligned}$$

Indeed, suppose that $i(H(1,.), U, \mathcal{K}) \neq 0$. Then there exists $x_1 \in U$ such that $Ax_1 + \lambda_0 v_0 = x_1$, which implies that $\lambda_0 \leq \|v_0\|^{-1}(\|x_1\| + \|Ax_1\|)$, contradicting (1.9). $\square$

Remark 1.9. *Letting $\theta = 0$, we obtain the computations of the index in case of a cone.*

Now, let Y be a retract of a Banach space E, Ω a subset of Y, and U a bounded open subset of Y. Assume that $T : \Omega \to E$ is a mapping such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$ and $F : \overline{U} \to E$ is a k-set contraction. Suppose that

$$0 \leq k < \gamma^{-1},$$

$$F(\overline{U}) \subset (I - T)(\Omega),$$

and

$$x \neq Tx + Fx, \quad \text{for all } x \in \partial U \bigcap \Omega.$$

Then $x \neq (I-T)^{-1}Fx$, for all $x \in \partial U$ and the mapping $(I-T)^{-1}F : \overline{U} \to Y$ is a strict γk-set contraction. Indeed, $(I-T)^{-1}F$ is continuous and bounded and for any bounded set B in U, we have

$$\alpha(((I-T)^{-1}F)(B)) \leq \gamma\, \alpha(F(B)) \leq \gamma k \alpha(B).$$

By Lemma 1.2, the fixed point index $i\left((I-T)^{-1}F, U, Y\right)$ is well defined. Thus we put

$$i_* \left(T+F, U \bigcap \Omega, Y\right) = \begin{cases} i\left((I-T)^{-1}F, U, Y\right), & U \bigcap \Omega \neq \emptyset \\ 0, & U \bigcap \Omega = \emptyset. \end{cases} \tag{1.10}$$

The proof of different results on the fixed point index i_* presented in this thesis invokes the following main properties of this index.

(a) (Normalization). If $Fx = y_0$, for all $x \in \overline{U}$, where $(I-T)^{-1}y_0 \in U \cap \Omega$, then

$$i_* \left(T+F, U \cap \Omega, Y\right) = 1.$$

(b) (Additivity). For any pair of disjoint open subsets $U_1, U_2 \subset U$ such that $T+F$ has no fixed point on $(\overline{U} \backslash (U_1 \cup U_2)) \cap \Omega$, we have

$$i_*(T+F, U \cap \Omega, Y) = i_*(T+F, U_1 \cap \Omega, Y) + i_*(T+F, U_2 \cap \Omega, Y).$$

(c) (Homotopy invariance). The fixed point index $i_*(T+H(t,.), U \cap \Omega, Y)$ does not depend on the parameter $t \in [0,1]$, where

(i) $H : [0,1] \times \overline{U} \to E$ is continuous and $H(t,x)$ is uniformly continuous in t with respect to $x \in \overline{U}$,

(ii) $H([0,1] \times \overline{U}) \subset (I-T)(\Omega)$,

(iii) $H(t,.) : \overline{U} \to E$ is a ℓ-set contraction with $0 \leq \ell < \gamma^{-1}$ for all $t \in [0,1]$,

(iv) $Tx + H(t,x) \neq x$ for all $t \in [0,1]$ and $x \in \partial U \cap \Omega$.

(d) (Solvability). If $i_*(T+F, U \cap \Omega, Y) \neq 0$, then $T+F$ has a fixed point in $U \cap \Omega$.

For more details about the definition of the index i_* and its properties see [34, 43].

Fixed point index on translates of cones for sum of operators

In [43], Mebarki and Georgiev generalized the fixed point index on translates of cones for the sum $T+F$ where T is such that $(I-T)$ is Lipschitz invertible map and F a k-set contraction. Let E be a Banach space, $\mathcal{P} \subset E$ ($\mathcal{P} \neq \{0\}$) be a cone. Given $\theta \in E$ ($\theta \neq 0$), consider the translate of $\mathcal{P}$, namely

$$\mathcal{K} = \mathcal{P} + \theta = \{x + \theta, \ \ x \in \mathcal{P}\}.$$

Let Ω be any subset of $\mathcal{K}$ and U a bounded open of $\mathcal{K}$ such that $U \bigcap \Omega \neq \emptyset$. We denote by $\overline{U}$ and ∂U the closure and the boundary of U relative to $\mathcal{K}$.

Since $\mathcal{K}$ is a closed convex of E, hence a retract, from (1.10) the fixed point index $i_*(T+F, U \bigcap \Omega, \mathcal{K})$ given by

$$i_*(T+F, U \bigcap \Omega, \mathcal{K}) = i\,((I-T)^{-1}F, U, \mathcal{K}), \tag{1.11}$$

is well defined whenever $T : \Omega \to E$ is a mapping such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$ and $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$ and $F(\overline{U}) \subset (I-T)(\Omega)$. In the following, we will calculate the fixed point index on translate of cones under various considerations. These calculations follow directly from the properties of this index.

Proposition 1.6. *Let U be a bounded open subset of $\mathcal{K}$ with $\theta \in U$. Assume that the mapping $T : \Omega \subset \mathcal{K} \to E$ is such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{U}) \subset (I-T)(\Omega)$. If*

$$Fx \neq (I-T)(\lambda x + (1-\lambda)\theta) \ \ \text{for all } x \in \partial U, \ \lambda \geq 1 \text{ and } \lambda x + (1-\lambda)\theta \in \Omega, \tag{1.12}$$

then $i_*(T+F, U \bigcap \Omega, \mathcal{K}) = 1$.

Proof. Define the homotopic deformation $H : [0,1] \times \overline{U} \to \mathcal{K}$ by

$$H(t,x) = t(I-T)^{-1}Fx + (1-t)\theta.$$

Then, the operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t,.)$ is a strict γk-set contraction for each t. Moreover, $H(t,.)$ has no fixed point on ∂U.

Otherwise, there would exist some $x_0 \in \partial U$ and $t_0 \in [0,1]$ such that $\frac{1}{t_0}x_0 + (1-\frac{1}{t_0})\theta \in \Omega$ for $t_0 \neq 0$, and

$$t_0(I-T)^{-1}Fx_0 + (1-t_0)\theta = x_0.$$

We may distinguish between two cases:

(i) If $t_0 = 0$, then $x_0 = \theta$, which is a contradiction.

(ii) If $t_0 \in (0,1]$, then $Fx_0 = (I-T)(\frac{1}{t_0}x_0 + (1-\frac{1}{t_0})\theta)$, which contradicts our assumption.

The properties of invariance by homotopy and normalization of the fixed point index guarantee that

$$i\left((I-T)^{-1}F, U, \mathcal{K}\right) = i\left(\theta, U, \mathcal{K}\right).$$

Consequently, by (1.11), we deduce that $i_*\left(T+F, U\bigcap\Omega, \mathcal{K}\right) = 1$. □

Proposition 1.7. *Let U be a bounded open subset of $\mathcal{K}$ with $\theta \in U$. Assume that the mapping $T : \Omega \subset \mathcal{K} \to E$ is such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$ and $F(\overline{U}) \subset (I-T)(\Omega)$. If*

$$\|Fx - T\theta - \theta\| \leq \|x-\theta\| \ \ \textit{and} \ \ Tx + Fx \neq x, \ \ \textit{for all } x \in \partial U \bigcap \Omega, \tag{1.13}$$

then $i_*\left(T+F, U\bigcap\Omega, \mathcal{K}\right) = 1$.

Proof. The mapping $(I-T)^{-1}F : \overline{U} \to \mathcal{K}$ is a strict γk-set contraction.

Since $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, for each $x \in \overline{U}$

$$\begin{aligned} \|(I-T)^{-1}Fx - \theta\| &= \|(I-T)^{-1}Fx - (I-T)^{-1}(I-T)\theta\| \\ &\leq \gamma\|Fx + T\theta - \theta\|. \end{aligned} \tag{1.14}$$

Therefore, from (1.14) and (1.13), we conclude that for all $x \in \partial U$, we get

$$\begin{aligned} \|(I-T)^{-1}Fx - \theta\| &\leq \gamma\|Fx + T\theta - \theta\| \\ &\leq \|x-\theta\|. \end{aligned}$$

The claim then follows from (1.7), which completes the proof. □

Proposition 1.8. *Assume that the mapping $T : \Omega \subset \mathcal{K} \to E$ is such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and*

$\big(tF(\overline{U})+(1-t)\theta\big)\subset(I-T)(\Omega)$ for all $t\in[0,1]$. If $(I-T)^{-1}\theta\in U$, and

$$(I-T)x\neq\lambda Fx+(1-\lambda)\theta\ \text{ for all } x\in\partial U\bigcap\Omega\ \text{ and } 0\leq\lambda\leq1, \tag{1.15}$$

then $i_*\left(T+F,U\bigcap\Omega,\mathcal{K}\right)=1$.

Proof. Define the homotopic deformation $H:[0,1]\times\overline{U}\to E$ by

$$H(t,x)=tFx+(1-t)\theta.$$

Then, the operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t,.)$ is a k-set contraction for each t. Moreover, $T+H(t,.)$ has no fixed point on $\partial U\bigcap\Omega$. Otherwise, there would exist some $x_0\in\partial U\bigcap\Omega$ and $t_0\in[0,1]$ such that

$$Tx_0+t_0Fx_0+(1-t_0)\theta=x_0,$$

then $x_0-Tx_0=t_0Fx_0+(1-t_0)\theta$, then we obtain a contradiction with the hypothesis (1.15). By invariance property and the normalization property of the fixed point index, we conclude that

$$\begin{aligned} i_*\left(T+F,U\bigcap\Omega,\mathcal{K}\right) &= i_*\left(T+\theta,U\bigcap\Omega,\mathcal{K}\right)\\ &= i\left((I-T)^{-1}\theta,U\bigcap\Omega,\mathcal{K}\right)=1. \end{aligned}$$

□

Remark 1.10. *Proposition 1.8 will be used to prove the existence of at least one positive solution of BVP (3.1) in chapter 3.*

Proposition 1.9. *Let U be a bounded open subset of $\mathcal{K}$. Assume that the mapping $T:\Omega\subset\mathcal{K}\to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma>0$, $F:\overline{U}\to E$ is a k-set contraction with $0\leq k<\gamma^{-1}$, and $F(\overline{U})\subset(I-T)(\Omega)$ for all $t\in[0,1]$. If there exists $u_0\in\mathcal{P}^*$ such that*

$$Fx\neq(I-T)(x-\lambda u_0),\qquad \text{for all } \lambda\geq0 \text{ and } x\in\partial U\cap(\Omega+\lambda u_0), \tag{1.16}$$

then the fixed point index $i_*(T+F,U\cap\Omega,\mathcal{K})=0$.

Proof. The mapping $(I-T)^{-1}F : U \to \mathcal{K}$ is a strict γk-set contraction. Suppose that $i_*(T+F, U\cap\Omega, \mathcal{K}) \neq 0$. Then,

$$i((I-T)^{-1}F, U, \mathcal{K}) \neq 0.$$

For each $r>0$, define the homotopy:

$$H(t,x) = (I-T)^{-1}Fx + tru_0, \text{ for } x \in \overline{U} \text{ and } t \in [0,1].$$

The operator H is continuous and uniformly continuous in t for each x. Moreover, $H(t,.)$ is a strict γk-set contraction for each t and

$$H([0,1]\times\overline{U}) = (I-T)^{-1}F(U) + tru_0 \subset \mathcal{K}.$$

We check that $H(t,x) \neq x$, for all $(t,x) \in [0,1]\times\partial U$. If $H(t_0,x_0) = x_0$ for some $(t_0,x_0) \in [0,1]\times\partial U$, then

$$x_0 - t_0ru_0 = (I-T)^{-1}Fx_0,$$

and so $x_0 - t_0ru_0 \in \Omega$. Hence

$$(I-T)(x_0 - t_0ru_0) = Fx_0,$$

for $x_0 \in \partial U \bigcap(\Omega + t_0ru_0)$, contradicting assumption (1.16). By homotopy invariance property of the fixed point index, we deduce that

$$i\left((I-T)^{-1}F + ru_0, U\bigcap\Omega, \mathcal{K}\right) = i\left((I-T)^{-1}F, U\bigcap\Omega, \mathcal{K}\right) \neq 0.$$

Thus the existence property of the fixed point index, for each $r>0$, there exists $x_r \in U$ such that

$$x_r - (I-T)^{-1}Fx_r = ru_0. \tag{1.17}$$

Letting $r \to +\infty$ in (1.17), the left hand side of (1.17) is bounded while the right hand side is not, which is a contradiction. Therefore

$$i_*\left(T+F, U\bigcap\Omega, \mathcal{K}\right) = 0,$$

which completes the proof. □

Remark 1.11. *We obtain the computation of the fixed point index on cones by setting $\theta = 0$.*

Part II

Difference equations

2 General results for linear difference equations

Difference equations appeared much earlier than differential equations, and since the invention of computers, difference equations have started receiving the attention they deserve, where differential equations are solved by using their difference equation formulations. Further, it was taken for granted that the theories of difference and differential equations are parallel. After the publication of the pioneer paper by Hartman [50] in the year 1978, a significant diversities and wide applications have made difference equations one of the major areas of research.

We begin this chapter with a number of examples that illustrate how difference equations arise in a variety fields of applications.

2.1 Examples and motivation

Example 2.1. *It is observed that the mass of a radioactive substance decrease over a fixed time period in a manner that is proportionate to the mass that was present at the beginning of the time period. If the half life of radium is* 1600 *years, find a formula for its mass as a function of time.*

Let $m(t)$ represent the mass of the radium after t years. Then

$$m(t+1) - m(t) = -km(t),$$

where k is a positive constant. Then

$$m(t+1) = (1-k)m(t), \ t \in \{0, 1, 2, \ldots\}.$$

Computing m recursively, we find

$$m(1) = m(0)(1-k),$$

$$m(2) = m(0)(1-k)^2,$$

$$\vdots$$

$$m(t) = m(0)(1-k)^t.$$

Since the half life of a radioactive substance is 1600,

$$m(1600) = m(0)(1-k)^{1600} = \frac{1}{2}m(0),$$

so

$$1-k = \left(\frac{1}{2}\right)^{\frac{1}{1600}}$$

Finally, we obtain

$$m(t) = m(0)\left(\frac{1}{2}\right)^{\frac{t}{1600}}$$

This problem is traditionally solved in calculus and physics textbooks by setting up and integrating the differential equation $m'(t) = -km(t)$. However, the solution presented here, using a difference equation, is somewhat shorter and employs only elementary algebra.

Example 2.2. *The Fibonacci sequence is $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, where each integer after the first two is the sum of the two integers immediately preceding it. Certain natural phenomena, such as the spiral patterns on sunflowers and pine cones, appear to be governed by this sequence. Let F_n denote the nth term in the Fibonacci sequence for $n = 1, 2, \ldots$. F_n is called the "nth Fibonacci number" and satisfies the initial value problem*

$$F_{n+2} - F_{n+1} - F_n = 0, \quad (n = 1, 2, \ldots)$$

$$F_1 = 1, \; F_2 = 1.$$

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$, so $\lambda = \frac{1\pm\sqrt{5}}{2}$. Then the general solution of the difference equation is

$$F_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

By using the initial conditions, we find $C_1 = -C_2 = \frac{1}{\sqrt{5}}$, *so*

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for $n = 1, 2, \ldots$. *Although* $\sqrt{5}$ *is predominant in this formula, all these numbers must be integers! Note that*

$$\frac{F_{n+1}}{F_n} = \frac{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}{1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n} \to \frac{1+\sqrt{5}}{2}$$

as $n \to \infty$. *The ratio* $\frac{1+\sqrt{5}}{2}$ *is known as the "golden section".*

Example 2.3. *Consider a channel for instance a telephone line, and suppose that two elementary information* S_1 *and* S_2 *of duration* k_1 *and* k_2 *respectively can be combined in order to obtain a message. Let* k *be a time interval greater than both* k_1 *and* k_2. *We are interested in the number of messages* $u(k)$ *of length* k, *we can distinguish two types of messages: those ending with* S_1 *and those ending with* S_2. *Then we have*

$$u(k) = u(k-k_1) + u(k-k_2), \tag{2.1}$$

where

$u(k-k_1)$ *is the number of messages ending with* S_1,

$u(k-k_2)$ *is the number of messages ending with* S_2.

Suppose for example that $k_1 = 1$ *and* $k_2 = 2$, *then the equation 2.1 becomes*

$$u(k) = u(k-1) + u(k-2),$$

with initial conditions $u(1) = 1$, $u(2) = 1$. *This initial value problem defines the Fibonacci numbers.*

Example 2.4. *Consider the following electric circuit*

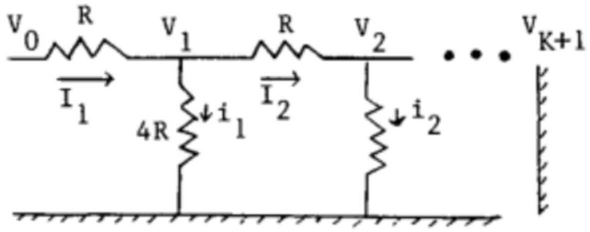

Figure 2.1

Assume that $V_0 = A$ *is a given voltage and* $V_{K+1} = 0$*, and the shaded region represent where the voltage is zero. Each resistance in the horizontal line is equal to* R *and in the vertical lines it is equal to* $4R$*.*

We want to find the voltage for $1 \leq k \leq K$*. For this, by using the Kirchoff's current law that state that the sum of the currents entering a junction point is equal the sum of the currents leaving the junction point. From the junction point corresponding to the voltage* V_{k+1}*, we obtain*

$$I_{k+1} = I_{k+2} + i_{k+1}.$$

Using Ohm's law $I = \frac{V}{R}$*, the above equation becomes*

$$\frac{V_k - V_{k+1}}{R} = \frac{V_{k+1} - V_{k+2}}{R} + \frac{V_{k+1} - 0}{4R}.$$

Identifying V_k *as* $u(k)$ *leads to the second order difference equation*

$$4u(k+2) - 9u(k+1) + 4u(k) = 0, \quad k \in \mathbb{N} \cap (0, K-1),$$

with boundary conditions

$$u(0) = A, \quad u(k+1) = 0.$$

Many of the calculations involved in solving and analyzing difference equations can be simplified by use of the difference calculus, a collection of mathematical tools quite similar to the differential calculus and just as the differential operator plays the central role in the differential calculus, the difference operator is the basic component of calculations involving finite differences. Difference calculus is the discrete analogue of the familiar differential and integral calculus. In the following sections we introduce some very basic properties of two operators that are essential in the study of difference equations. These operators are the difference operator and the summation operator.

2.2 Difference operator

Definition 2.1. *Let* y *be a function of a real variable* t*. The difference operator* Δ *is defined by:* $\Delta y(t) = y(t+1) - y(t)$*.*

Remark 2.1. *Occasionally we will apply the difference operator to a function of two or more variables. In this case, a subscript will be used to indicate which variable is to be shifted by one unit. For example,*

$$\Delta_t te^n = (t+1)e^n - te^n = e^n,$$

while

$$\Delta_n te^n = te^{n+1} - te^n = te^n(e-1).$$

An elementary operator that is often used in conjunction with the difference operator is the shift operator.

Definition 2.2. *The shift operator E is defined by: $Ey(t) = y(t+1)$.*

Remark 2.2. *1. Δ and E are linear operators.*

2. Δ and E commute, i.e. $\Delta E = E\Delta$.

3. $\Delta = E - I$ where I denotes the identity operator.

Definition 2.3. *Let $E^0 = \Delta^0 = I$. We define Δ^n and E^n for $n \in \mathbb{N}^*$ respectively by:*

(i) $\Delta^n y(t) = \Delta(\Delta^{n-1}y(t))$.

(ii) $E^n y(t) = y(t+n)$.

Lemma 2.1. *Using the Binomial Theorem from algebra we obtain:*

(i) $\Delta^n y(t) = (E-I)^n y(t) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k E^{n-k} y(t)$.

(ii) $E^n y(t) = (\Delta + I)^n y(t) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} y(t)$.

Proposition 2.1. *[56, Theorem 2.1]*

(a) $\Delta^m(\Delta^n y(t)) = \Delta^{m+n} y(t)$ *for all positive integers m and n.*

(b) $\Delta(y(t) + z(t)) = \Delta y(t) + \Delta z(t)$.

(c) $\Delta(c\,y(t)) = c\,\Delta y(t)$, *if c is a constant.*

(d) $\Delta(y(t)z(t)) = y(t)\Delta z(t) + Ez(t)\Delta y(t)$.

(e) $\Delta(\frac{y(t)}{z(t)}) = \frac{z(t)\Delta y(t) - y(t)\Delta z(t)}{z(t)Ez(t)}$.

Proposition 2.2. *Let $\mathbb{N}_{n_0} = \{n > n_0, \quad n_0 \in \mathbb{N}\}$, the following statements hold:*

(a) $\sum\limits_{i=n_0}^{n-1} \Delta x(i) = x(n) - x(n_0), \quad n \in \mathbb{N}_{n_0}$.

(b) $\Delta\left(\sum\limits_{i=n_0}^{n-1} x(i)\right) = x(n), \quad n \in \mathbb{N}_{n_0}$.

(c) *Let $P(n) = \sum\limits_{i=0}^{k} a_i n^{k-i}$ be a polynomial of degree k, where $\{a_0, a_1, \dots, a_k\}$ are constants. Then*

$$\Delta^k P(n) = a_0 k! \tag{2.2}$$

$$\Delta^{k+i} P(n) = 0, \quad \forall i \geq 1. \tag{2.3}$$

Proof. Using the definition of Δ, we obtain

(a)

$$\begin{aligned}\sum_{i=n_0}^{n-1} \Delta x(i) &= \sum_{i=n_0}^{n-1} (x(i+1) - x(i)) \\ &= \sum_{i=n_0+1}^{n} x(i) - \sum_{i=n_0}^{n-1} x(i) \\ &= x(n) - x(n_0).\end{aligned}$$

(b)

$$\begin{aligned}\Delta\left(\sum_{i=n_0}^{n-1} x(i)\right) &= \sum_{i=n_0}^{n} x(i) - \sum_{i=n_0}^{n-1} x(i) \\ &= x(n).\end{aligned}$$

(c) On the one hand we have

$$\Delta p(n) = \sum_{i=0}^{k} a_i (n+1)^{k-i} - \sum_{i=0}^{k} a_i n^{k-i}.$$

On the other hand, we have

$$\begin{aligned}(n+1)^k &= \sum_{i=0}^{k} \binom{k}{i} n^i = 1 + kn + \frac{k(k-1)}{2} n^2 + \cdots + kn^{k-1} + n^k \\ (n+1)^{k-1} &= 1 + (k-1)n + \frac{(k-1)(k-2)}{2} n^2 + ... + (k-1)n^{k-2} + n^{k-1} \\ &\vdots\end{aligned}$$

Then $\Delta p(n) = a_0 k n^{k-1} + P_1(n)$, where P_1 is a polynomial of degree strictly less than $k-1$. Similarly, we can show that

$$\begin{aligned} \Delta^2 p(n) &= a_0 k(k-1)n^{k-2} + P_2(n), \text{ where } deg(P_2) < k-2, \\ \Delta^3 p(n) &= a_0 k(k-1)(k-2)n^{k-3} + P_3(n), \text{ where } deg(P_3) < k-3, \\ &\vdots \end{aligned}$$

Carrying out this process k times we obtain the equality (2.2).

To show (2.3) we use the definition of $\triangle$ and (2.2). □

Proposition 2.3. *[56, Theorem 2.2] Let a be a constant. Then*

(a) $\Delta\, a^t = (a-1)a^t$.

(b) $\Delta \sin at = 2\sin\frac{a}{2}\cos a(t+\frac{1}{2})$.

(c) $\Delta \cos at = -2\sin\frac{a}{2}\sin a(t+\frac{1}{2})$.

(d) $\Delta \ln at = \ln(1+\frac{1}{t})$.

(e) $\Delta \ln\Gamma(t) = \ln t$.

Here $\ln t$ represents any logarithm of the positive number t.

Remark 2.3. *1. One of the most basic special formulas in the differential calculus is the power rule $\frac{d}{dt}t^n = nt^{n-1}$. Unfortunately, the difference of a power is complicated and, as a result, is not very useful:*

$$\begin{aligned} \Delta_t t^n &= (t+1)^n - t^n \\ &= \sum_{k=0}^{n}\binom{n}{k}t^k - t^n \\ &= \sum_{k=0}^{n-1}\binom{n}{k}t^k. \end{aligned}$$

2. All the formulas in Proposition 2.3 remain valid if a constant "shift" is introduced in the t variable. For example, $\Delta a^{t+k} = (a-1)a^{t+k}$.

3. The formulas in Propositions 2.1 and 2.3 can be used in combination to find the differences of more complicated expressions. However, it may be easier to use the definition directly.

2.3 Summation operator

In this section, we introduce the right inverse operator of the difference operator, which is called the "Indefinite sum" or "Antidifference".

Definition 2.4. *An indefinite sum (or antidifference) of $y(t)$, denoted $\sum y(t)$, is any function so that $\Delta(\sum y(t)) = y(t)$ for t in the domain of y.*

Remark 2.4.
1. *Recall that the indefinite integral plays a similar role in the differential calculus.*
2. *The indefinite sum is also not unique as the indefinite integral.*

Theorem 2.1. *[56, Theorem 2.4] If $z(t)$ is an indefinite sum of $y(t)$, then every indefinite sum of $y(t)$ is given by $\sum y(t) = z(t) + c(t)$, where c has the same domain as y and $\Delta c(t) = 0$.*

Example 2.5. *Compute the indefinite sum $\sum 6^t$.*

From the Proposition 2.3, $\Delta 6^t = 5.6^t$, so we have $\Delta \frac{6^t}{5} = 6^t$.

It follows that $\frac{6^t}{5}$ is an indefinite sum of 6^t and we write:

$$\sum 6^t = \frac{6^t}{5} + c(t),$$

where $c(t)$ is any function with the same domain as 6^t and $\Delta c(t) = 0$.

Corollary 2.1. *Let y be defined on a set of the type $\{a, a+1, a+2, \ldots\}$, where a is any real number, and let $z(t)$ be an indefinite sum of $y(t)$. Then every indefinite sum of $y(t)$ is given by $\sum y(t) = z(t) + c$, where c is an arbitrary constant.*

Remark 2.5.
1. *If the domain of y is a set of integer. Then $\Delta c(t) = c(t+1) - c(t) = 0$, that is, $c(1) = c(2) = c(3) = \ldots$, so $c(t)$ is a constant.*
   In this case, we write $\sum 6^t = \frac{6^t}{5} + c$, where c is any constant.
2. *If the domain of y is the set of all real numbers, then the equation $\Delta c(t) = c(t+1) - c(t) = 0$ says that $c(t+1) = c(t)$ for all real t, which means that c can be any periodic function having period one. For example, we could choose $c(t) = 2\sin 2\pi t$, or $c(t) = -5\cos 4\pi(t-\pi)$, in Theorem 2.1 and obtain an indefinite sum.*

Proposition 2.4. *[56, Theorem 2.6] Every indefinite sum satisfies the following properties:*

(a) $\sum(y(t) + z(t)) = \sum y(t) + \sum z(t)$.

(b) $\sum(\lambda\, y(t)) = \lambda \sum y(t)$ *if* λ *is a constant.*

(c) $\sum(y(t)\, \Delta z(t)) = y(t)\, z(t) - \sum(Ez(t)\, \Delta y(t))$.

(d) $\sum(Ey(t)\, \Delta z(t)) = y(t)\, z(t) - \sum z(t)\, \Delta y(t)$.

Remark 2.6.

1. *The formulas (c) and (d) in the above Theorem are called summation by parts formulas.*

2. *The summation by parts formulas can be used to compute certain indefinite sums much as the integration by parts formula is used to compute integrals.*

Proposition 2.5. *[56, Theorem 2.5] Let* a *be a constant. Then, for* $\Delta c(t) = 0$, *we have*

(a) $\sum a^t = \frac{a^t}{a-1} + c(t),\ (a \neq 1)$.

(b) $\sum \sin at = -\frac{\cos a(t-\frac{1}{2})}{2\sin\frac{a}{2}} + c(t),\ (a \neq 2n\pi)$.

(c) $\sum \cos at = \frac{\sin a(t-\frac{1}{2})}{2\sin\frac{a}{2}} + c(t),\ (a \neq 2n\pi)$.

(d) $\sum \ln t = \ln\Gamma(t) + c(t),\ (t > 0)$.

(e) $\sum \binom{t}{a} = \binom{t}{a+1} + c(t)$.

(f) $\sum \binom{a+t}{t} = \binom{a+t}{t-1} + c(t)$.

Remark 2.7. *The operators* Δ *and* $\sum$ *do not commute, such as in the differential calculus the operators* D *(derivative) and* $\int$ *do not commute.*
For example, $\sum 1 = t$.

$$\Delta t = (t+1) - t = 1.$$

Apply $\sum$ *to this equation, then* $t + c(t) = \sum 1$, *where* $c(t)$ *is an arbitrary periodic function of period* 1.

Proposition 2.6. **(a)** *Let $m, n, p \in \mathbb{N}$. For m fixed and $n \geq m$, we have $\Delta\left(\sum\limits_{k=m}^{n-1} y_k\right) = y_n$, then,*

$$\sum y_n = \sum_{k=m}^{n-1} y_k + c, \text{ where } c \text{ is a constant.}$$

(b) *For p fixed and $p \geq n$, we have $\Delta\left(\sum\limits_{k=n}^{p} y_k\right) = -y_n$, then,*

$$\sum y_n = -\sum_{k=n}^{p} y_k + d, \text{ where } d \text{ is a constant.}$$

Proposition 2.7. *[56, Theorem 2.7] If z_n is an indefinite sum of y_n and $m, n\,(n \geq m)$ are integers, then*

$$\sum_{k=m}^{n-1} y_k = [z_n]_m^n = z_n - z_m.$$

2.4 General results for linear difference equations

This section investigates the essential techniques employed in the treatment of linear difference equations. We begin Section 2.4 with the fundamental theory of linear difference equations, and then we develop the method of variation of constants. We then specialize our discussions on linear difference equations with constant coefficients, since this is an important class in itself.

First order difference equations

Definition 2.5. *A first order linear difference equation is an equation in the form:*

$$y(t+1) - p(t)y(t) = r(t), \tag{2.4}$$

where $p(t)$ and $r(t)$ be given functions with $p(t) \neq 0$ for all t.

If $p(t) = 1$ for all t, then equation (2.4) is simply

$$\Delta y(t) = r(t),$$

its solution is

$$y(t) = \sum r(t) + c(t),$$

where $\Delta c(t) = 0$.

Consider the first order homogeneous equation

$$u(t+1) = p(t)u(t), \quad t \in \{a, a+1, \ldots\},\ a \in \mathbb{R} \tag{2.5}$$

which is easily solved by iteration:

$$\begin{aligned} u(a+1) &= p(a)u(a) \\ u(a+2) &= p(a+1)p(a)u(a) \\ &\vdots \\ u(a+n) &= u(a)\prod_{k=0}^{n-1} p(a+k). \end{aligned}$$

Then

$$u(t) = u(a)\prod_{s=a}^{t-1} p(s),\ (t = a, a+1, ...),$$

where it is understood that $\prod\limits_{s=a}^{a-1} p(s) = 1$.

Equation (2.4) can be solved by substituting $y(t) = u(t)v(t)$, where v is to be determined:

$$u(t+1)v(t+1) - p(t)u(t)v(t) = r(t),$$

$$u(t+1)v(t+1) + v(t)[u(t+1) - p(t)u(t)] - v(t)u(t+1) = r(t),$$

$$u(t+1)[v(t+1) - v(t)] = r(t),$$

or

$$Eu(t)\Delta v(t) = r(t),$$

or

$$\Delta v(t) = \frac{r(t)}{Eu(t)}.$$

So,

$$v(t) = \sum \frac{r(t)}{Eu(t)} + c.$$

Then,

$$y(t) = u(t)\left[\sum \frac{r(t)}{Eu(t)} + c\right].$$

The last equation with c an arbitrary constant gives us a representation of all solutions of the equation (2.4) provided u is any nontrivial solution of the equation (2.5).

Linear difference equations of order $k \geq 1$

Definition 2.6. *An equation in the form:*

$$y(t+k) + p_1(t)\,y(t+k-1) + \ldots + p_k(t)\,y(t) = f(t), \tag{2.6}$$

where $p_1(t), \ldots, p_k(t)$ and $f(t)$ are assumed to be known and $p_k(t) \neq 0$, for all t, is called linear non-homogeneous difference equations of order k.

Definition 2.7. *The homogeneous corresponding equation to (2.6) is given by*

$$y(t+k) + p_1(t)\,y(t+k-1) + \ldots + p_k(t)\,y(t) = 0, \tag{2.7}$$

Remark 2.8.

1. *The equation (2.6) can also be written using the shift operator as*

$$(E^k + p_1(t)E^{k-1} + \ldots + p_k(t)E^0)y(t) = f(t), \quad \textit{where } E^0 = I.$$

2. *Since $E = \Delta + I$, it is also possible to write equation (2.6) in terms of the difference operator.*

Definition 2.8. *If we specify k initial conditions of the equation (2.6), we are led to the following corresponding initial value problem, where the c_i, $i \in \{0, \ldots, k-1\}$ are real numbers*

$$y(t+k) + p_1(t)\,y(t+k-1) + \ldots + p_k(t)\,y(t) = f(t), \quad t \in \{a, a+1, \ldots\}, \tag{2.8}$$

$$y(t_0) = c_0,\, y(t_0+1) = c_1,\, \ldots, y(t_0+k-1) = c_{k-1}, \quad \forall\, t_0 \in \{a, a+1, \ldots\}. \tag{2.9}$$

Theorem 2.2. *[35, Theorem 2.7, page 66] The initial value problem (2.8)-(2.9) have a unique solution.*

Example 2.6. *Consider the following difference equation of second order*

$$y(n+2) + \frac{n}{n+1}\,y(n+1) - 3\,y(n) = n,\ n \in \mathbb{N},$$

with $y(0)$=1 and $y(1) = -1$ and we will find the values of $y(3)$ and $y(4)$.

First we rewrite the equation in the convenient form

$$y(n+2) = -\frac{n}{n+1}\,y(n+1) + 3y(n) + n,\ n \in \mathbb{N}. \tag{2.10}$$

Letting $n = 0$ in (2.10), we have

$$y(2) = 3y(0) = 3.$$

For $n = 1$, we have

$$y(3) = -\frac{1}{2}y(2) + 3y(1) + 1 = -\frac{7}{2}\cdot$$

Then,

$$y(4) = -\frac{2}{3}y(3) + 3y(2) + 1 = \frac{37}{3}\cdot$$

Theorem 2.3. *[56, Theorem 3.3, page 51]*

(a) *If y_1 and y_2 are solutions to the homogenous equation (2.7), then $c_1y_1 + c_2y_2$ with c_1, c_2 are constant solve the equation (2.7).*

(b) *If y_1 is a solution to the equation (2.7) and y_2 is a solution to the equation (2.6), then y_1+y_2 solve the non homogenous equation (2.6).*

(c) *If y_1 and y_2 are solutions to the equation (2.6), then $y_1 - y_2$ solves the homogenous equation (2.7).*

Remark 2.9. *Solutions of the non homogeneous equation (2.6) do not form a vector space. In particular, neither the sum(difference) of two solutions nor a multiple of a solution is a solution.*

Corollary 2.2. *If z is a solution of equation (2.6), then every solution y of equation (2.6) takes the form $y = z + u$ where u is some solution of equation (2.7).*

Lemma 2.2. *Define the operator L by*

$$L(y) = \sum_{i=0}^{k} p_i(t)y(t + k - i), \; t \in \mathbb{R}. \tag{2.11}$$

Then, L is linear.

Proof. Let $\alpha, \beta \in \mathbb{R}$ and $t \in \mathbb{R}$, then

$$\begin{aligned} L(\alpha x(t) + \beta y(t)) &= \sum_{i=0}^{k} p_i(t)(\alpha x(t + k - i) + \beta y(t + k - i)) \\ &= \alpha \sum_{i=0}^{k} p_i(t)x(t + k - i) + \beta \sum_{i=0}^{k} p_i(t)y(t + k - i) \\ &= \alpha Lx(t) + \beta Ly(t). \end{aligned}$$

□

Remark 2.10. *Let L defined by (2.11), then equation (2.6) takes the form*

$$Ly(t) = f(t), \ t \in \mathbb{R} \tag{2.12}$$

and equation (2.7) will be

$$Ly(t) = 0, \ t \in \mathbb{R}, \tag{2.13}$$

with $p_0(t) = 1$.

Proposition 2.8. *Let S be the set of solutions of equation (2.13). Then S is a vector space $\mathbb{R}$.*

Proof. Direct consequence of lemma 2.2. In fact, S is the kernel of the linear operator L. □

Definition 2.9. *The set of functions $\{y_1, y_2, \ldots, y_k\}$ is linearly dependent on the set $t \in \{a, a+1, \ldots\}$ if there are constants $c_1, \ldots, c_k$, not all zero, so that*

$$c_1y_1(t) + c_2y_2(t) + \ldots + c_ky_k(t) = 0, \qquad \textit{for } t \in \{a, a+1, \ldots\}.$$

Otherwise the set is linearly independent.

Theorem 2.4. *Let $\{y_1, y_2, \ldots, y_k\}$ be the set of linearly independent solutions of equation (2.13) and let y_p be a particular solution of equation (2.12), then any other solution y of non-homogeneous equation (2.12) can be written in the form*

$$y(t) = \sum_{i=1}^{k} c_iy_i(t) + y_p(t), \ c_i \in \mathbb{R}, \ i \in \{1, \ldots, k\} \ \textit{and} \ t \in \mathbb{R}. \tag{2.14}$$

Proof. Let y be a solution of equation (2.12) et y_p a particular solution of the same equation. From lemma 2.2, the function $(y - y_p)$ is a solution to equation (2.13). Then

$$y(t) - y_p(t) = \sum_{i=1}^{k} c_iy_i(t), \ t \in \mathbb{R}. \tag{2.15}$$

Hence

$$y(t) = \sum_{i=1}^{k} c_iy_i(t) + y_p(t), \ t \in \mathbb{R}. \tag{2.16}$$

□

Definition 2.10. *A set of k linearly independent solutions of (2.12) is called a fundamental set of solutions.*

It is not practical to check the linear independence of a set of solutions using the definition. Fortunately, there is a simple method to check the linear independence of solutions using the Casoratian.

Definition 2.11. *Let $\{y_1, y_2, \ldots, y_k\}$ be the set of solution of the equation (2.13). The Casoratian of these solutions is given by:*

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_k(t) \\ y_1(t+1) & y_2(t+1) & \ldots & y_k(t+1) \\ \vdots & \vdots & \ddots & \vdots \\ y_1(t+k-1) & y_2(t+k-1) & \ldots & y_k(t+k-1) \end{vmatrix}$$

Remark 2.11. *The Casoratian is the discrete analogue of the Wronskian in differential equations.*

Example 2.7. *$\{t, 2^t\}$ is a fondamental set of solutions of the equation :*

$$y(t+2) - \frac{3t-2}{t-1} y(t+1) + \frac{2t}{t-1} y(t) = 0. \tag{2.17}$$

$$W(t) = \begin{vmatrix} t & 2^t \\ t+1 & 2^t+1 \end{vmatrix} = -1 \neq 0$$

Hence by Theorem 2.5, the solutions $t, 2^t$ are linearly independent and thus form a fundamental set of solutions for (2.17).

Remark 2.12. *It is not difficult to check that the Casoratian satisfies the equation*

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_k(t) \\ \Delta y_1(t) & \Delta y_2(t) & \ldots & \Delta y_k(t) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{k-1} y_1(t) & \Delta^{k-1} y_2(t) & \ldots & \Delta^{k-1} y_k(t) \end{vmatrix}$$

Lemma 2.3. ***Abel's lemma****[35, Theorem 2.13, page 68]*

Let $y_1, y_2, \ldots, y_k$ be solutions of (2.7) and let W be their Casoratian. Then, for $n \geq n_0$,

$$W(n) = (-1)^{k(n-n_0)} \left(\prod_{i=n_0}^{n-1} P_k(i) \right) W(n_0). \tag{2.18}$$

Corollary 2.3. *Suppose that $P_k(n) \neq 0$ for all $n \geq n_0$. Then the Casoratian $W(n) \neq 0$ for all $n \geq n_0$ if and only if $W(n_0) \neq 0$.*

Proof. This corollary follows immediately from formula (2.18). □

Theorem 2.5. *[56, Theorem 3.4, page 52] Let $y_1(t), \ldots, y_k(t)$ be solutions of equation (2.7) for $t \in \{a, a+1, \ldots\}$. Then the following statements are equivalent:*

(a) *The set $\{y_1(t), \ldots, y_k(t)\}$ is linearly dependent for $t \in \{a, a+1, \ldots\}$.*

(b) *$W(t) = 0$ for some t.*

(c) *$W(t) = 0$ for all t.*

Theorem 2.6. *(The Fundamental Theorem)*

If $P_k(t) \neq 0$ for all $t \geq t_0$, then (2.7) has a fundamental set of solutions for $t \geq t_0$.

Proof. By Theorem 2.2, the problem

$$y_i(t+k) + P_1(t)\, y_i(t+k-1) + \ldots + P_k(t)\, y_i(t) = 0,$$

$$y_i(t_0 + i - 1) = 1,\ y_i(t_0 + j) = 0, \quad \forall\, j \neq i-1,\ 1 \leq i \leq k,$$

has solutions $y_1, y_2, \ldots, y_k$. Let $W(t)$ be their Casoratian, then

$$W(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \ldots & y_k(t_0) \\ y_1(t_0+1) & y_2(t_0+1) & \ldots & y_k(t_0+1) \\ \vdots & \vdots & \ddots & \vdots \\ y_1(t_0+k-1) & y_2(t_0+k-1) & \ldots & y_k(t_0+k-1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{vmatrix} = 1.$$

It follows that $W(t_0) = 1 \neq 0$.

This implies that the set $\{y_1, y_2, \ldots, y_k\}$ is a fundamental set of solutions of equation (2.7). □

Theorem 2.7. *[56, Theorem 3.5, page 53] If $y_1, \ldots, y_k$ are independent solutions of the homogenous equation (2.7) then every solution y of equation (2.7) can be written in the form $y = c_1 y_1 + \ldots + c_k y_k$, for some constant $c_1, \ldots, c_k$.*

Linear difference equations with constant coefficient

Consider the following linear difference equations with constant coefficient of k^{th} order.

$$y(t+k) + p_1\, y(t+k-1) + p_2\, y(t+k-2) + \ldots + p_k\, y(t) = 0, \tag{2.19}$$

where $p_1, \ldots, p_k$ are constants and $p_k \neq 0$. Our goal in this paragraphe is to find a fondamental set of solutions and consequently the general solution of equation (2.19).

Definition 2.12. **(a)** *The polynomial $\lambda^k + p_1\lambda^{k-1} + \ldots + p_k$ is called the characteristic polynomial of equation (2.19).*

(b) *The equation $\lambda^k + p_1\lambda^{k-1} + \ldots + p_k = 0$ is the characteristic equation for the equation (2.19).*

(c) *The solutions $\lambda_1, \ldots, \lambda_k$ of the characteristic equation are the characteristic roots.*

Remark 2.13. *1. If we introduce the shift operator E into equation (2.19), it takes on the form of its characteristic equation and has similar factors:*

$$(E^k + p_{k-1}E^{k-1} + \ldots + p_k E^0)y(t) = 0,$$

or

$$(E-\lambda_1)^{m_1} \ldots (E-\lambda_r)^{m_r} y(t) = 0,$$

where $m_1 + m_2 + \ldots + m_r = k$.

2. Since $p_k \neq 0$, none of the characteristic roots is equal to zero.

Proposition 2.9. *Let λ be a real number different from zero, if $y(t) = \lambda^t, \quad \forall t \in \mathbb{R}$ is a solution to the equation (2.19), then λ is a solution to the equation*

$$\lambda^k + p_1\lambda^{k-1} + \ldots + p_k = 0. \tag{2.20}$$

Theorem 2.8. *[35, page 75] Let $\lambda_i, i \in \{1, \ldots, k\}$ distinct characteristic roots of the equation (2.20). Then $\{\lambda_1^t, \ldots, \lambda_k^t\}$ is a fundamental set of solutions of the equation (2.19).*

Proof. To prove this theorem, it suffices to show that $W(0) \neq 0$, where $W(t)$ is the Casoratian of the solutions. That is

$$W(0) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_1 & \dots & \lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \end{vmatrix}.$$

This determinant is called the Vandermonde determinant given by

$$W(0) = \prod_{1 \leq i < j \leq k} (\lambda_j - \lambda_i).$$

Since all the λ_i are distinct, then $W(0) \neq 0$. □

Corollary 2.4. *The general solution of equation* (2.19) *is given by*

$$y(t) = \sum_{i=1}^{k} C_i \lambda_i^t, \quad C_i \in \mathbb{R}.$$

Theorem 2.9. *[56, Theorem 3.6, page 55]*

Suppose that equation (2.19) *has characteristic roots* $\lambda_1, \lambda_2, \dots, \lambda_r$ *with multiplicities* $m_1, m_2, \dots, m_r$*, respectively with* $r \leq k$ *and* $m_1 + m_2 + \dots + m_r = k$*. Then*

$$\{\lambda_1^t,\ t\lambda_1^t,\ \dots,\ t^{m_1-1}\lambda_1^t,\ \lambda_2^t,\ t\lambda_2^t,\ \dots,\ t^{m_2-1}\lambda_2^t,\ \lambda_r^t,\ t\lambda_r^t,\ \dots,\ t^{m_r-1}\lambda_r^t\}$$

is a fundamental set of solutions of the equation (2.19).

Corollary 2.5. *The general solution of equation* (2.19) *can be written in the form*

$$y(t) = \sum_{i=1}^{r} \sum_{j=0}^{m_i-1} c_{i,j} t^j \lambda_i^t, \quad c_{i,j} \in \mathbb{R}.$$

Solving linear difference equations: Annihilator method

In this paragraphe we focus our attention on solving the k^{th} order linear non-homogeneous equation with constants coefficients

$$y(t+k) + p_1\, y(t+k-1) + \dots + p_k\, y(t) = f(t), \tag{2.21}$$

where $p_1, \dots, p_{k-1}$ are constants and $p_k \neq 0$.

The corresponding homogeneous equation is given by

$$y(t+k) + p_1\, y(t+k-1) + \dots + p_k\, y(t) = 0, \quad t \in \mathbb{R}. \tag{2.22}$$

Theorem 2.10. *[35, Theorem 2.30, page 84] Let $\{y_1, y_2, \dots, y_k\}$ be a fundamental set of solutions of the homogeneous equation(2.22) and y_p a particular solution of the non-homogeneous equation (2.21). Then any solution y of (2.21) may be written as*

$$y(t) = y_p(t) + \sum_{i=1}^{k} c_i y_i(t), \quad c_i \in \mathbb{R}.$$

Definition 2.13. *A polynomial operator $N(E)$, where E is the shift operator defined in Definition 2.2, is said to be an annihilator of f if*

$$N(E)\, f(t) = 0, \quad \forall t \in \mathbb{R}. \tag{2.23}$$

Let us now rewrite (2.21) using the shift operator E as

$$P(E)\, y(t) = f(t), \tag{2.24}$$

where $P(E) = E^{\,k} + p_1 E^{\,k-1} + \ldots + p_k I$.

Assume now that $N(E)$ is an annihilator of f in (2.24). Applying $N(E)$ on both sides of (2.24) yields

$$N(E)P(E)\, y(t) = 0, \quad \forall\, t \in \mathbb{R}. \tag{2.25}$$

- Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the characteristic roots of the homogeneous equation

$$P(E)y(t) = 0. \tag{2.26}$$

- Let $\mu_1, \mu_2, \dots, \mu_k$ be the characteristic roots of the equation

$$N(E)y(t) = 0. \tag{2.27}$$

We must consider two separate cases:

Case 1: $\lambda_i \neq \mu_j$, $i, j \in \{1, 2, \dots, k\}$. In this case, write the particular solution y_p in the form of the general solution of (2.27) with undetermined constants. Substituting back this particular solution into (2.21), we find the values of the constants.

Case 2: $\exists\, i, j \in \{1, 2, \dots, k\}$ such that $\lambda_i = \mu_j$. In this case, to determine a particular solution y_p, we first find the general solution of (2.25) and then drop all the terms that appear in the general solution of the homogenous equation associeted to (2.21). Then proceed as in case 1 to evaluate the constants.

Solving linear difference equations: Variation of parameters method

Consider the following difference equation of second order

$$p_0(t)y(t+2) + p_1(t)y(t+1) + p_2(t)y(t) = f(t), \quad t \in \mathbb{R} \tag{2.28}$$

where $p_0(t), p_1(t), p_2(t)$ and $f(t)$ are assumed to be known and $p_0(t) \neq 0, p_2(t) \neq 0$, for all $t \in \mathbb{R}$. The homogeneous equation corresponding to (2.28) is given by

$$p_0(t)y(t+2) + p_1(t)y(t+1) + p_2(t)y(t) = 0. \tag{2.29}$$

Let u_1, u_2 be two independent solutions of equation (2.29).

We seek a solution of equation (2.28) of the form $y(t) = a_1(t)u_1(t) + a_2(t)u_2(t)$, where a_1, a_2 are to be determined. Then

$$\begin{aligned} y(t+1) &= a_1(t+1)u_1(t+1) + a_2(t+1)u_2(t+1) &(2.30)\\ &= a_1(t)u_1(t+1) + a_2(t)u_2(t+1) + \Delta a_1(t)u_1(t+1) + \Delta a_2(t)u_2(t+1). &(2.31) \end{aligned}$$

In this method we choose a_1, a_2 so that

$$\Delta a_1(t)u_1(t+1) + \Delta a_2(t)u_2(t+1) = 0. \tag{2.32}$$

Next, from (2.30) and (2.32) we have

$$\begin{aligned} y(t+2) &= a_1(t+1)u_1(t+2) + a_2(t+1)u_2(t+2) &(2.33)\\ &= a_1(t)u_1(t+2) + a_2(t)u_2(t+2) + \Delta a_1(t)u_1(t+2) + \Delta a_2(t)u_2(t+2). &(2.34) \end{aligned}$$

Now substitue the expressions (2.30), (2.32) and (2.33) into (2.28) to obtain

$$\begin{aligned} p_0(t)y(t+2) + p_1(t)y(t+1) + p_2(t)y(t) &= a_1(t)[p_0(t)u_1(t+2) + p_1(t)u_1(t+1) + p_2(t)u_1(t)]\\ &\quad a(t)[p_0(t)u_2(t+2) + p_1(t)u_2(t+1) + p_2(t)u_2(t)]\\ &\quad +p_0(t)[u_1(t+2)\Delta a_1(t) + u_2(t+2)\Delta a_2(t)] \end{aligned}$$

Since u_1 and u_2 satisfy the equation (2.29) the first two bracketed expressions are zero. Then $y(t)$ satisfies equation (2.28) if

$$u_1(t+2)\Delta a_1(t) + u_2(t+2)\Delta a_2(t) = \frac{f(t)}{p_0(t)}. \tag{2.35}$$

Let $W(t)$ be the Casoratien of solutions u_1 and u_2, then

$$W(t+1) = \begin{vmatrix} u_1(t+1) & u_2(t+1) \\ u_1(t+2) & u_2(t+2) \end{vmatrix} = u_1(t+1)u_2(t+2) - u_2(t+1)u_1(t+2)$$

By multiplying (2.35) by $u_2(t+1)$ we obtain

$$u_1(t+2)u_2(t+1)\Delta a_1(t) + u_2(t+2)u_2(t+1)\Delta a_2(t) = \frac{f(t)}{p_0(t)}u_2(t+1). \tag{2.36}$$

Combining (2.32) with (2.36) we obtain

$$\Delta a_1(t)\left[u_1(t+2)u_2(t+1) - u_2(t+2)u_1(t+1)\right] = \frac{f(t)}{p_0(t)}u_2(t+1), \tag{2.37}$$

which implies

$$\Delta a_1(t) = -\frac{f(t)u_2(t+1)}{p_0(t)W(t+1)}, \quad t \in \mathbb{N}_0.$$

Then using the Proposition 2.2

$$a_1(t) = \sum_{i=0}^{t-1} -\frac{f(i)u_2(i+1)}{p_0(i)W(i+1)}, \quad t \in \mathbb{N}_0.$$

In the same way we get

$$\Delta a_2(t) = -\frac{f(t)u_1(t+1)}{p_0(t)W(t+1)}, \quad t \in \mathbb{N}_0.$$

Then

$$a_1(t) = \sum_{i=0}^{t-1} -\frac{f(i)u_1(i+1)}{p_0(i)W(i+1)}, \quad t \in \mathbb{N}_0.$$

2.5 The self-adjoint second order linear equation

In this section we will introduce the second order self-adjoint difference equation and we will show which second order linear difference equations can be put in the self-adjoint form. The linear second order self-adjoint difference equation is defined to be

$$\Delta(p(t-1)\Delta y(t-1)) + q(t)y(t) = 0, \quad t \in [a, b+1] = \{a, a+1, \dots, b+1\}, \tag{2.38}$$

where $p(t) > 0$ is defined on the set of integers $[a, b+1]$ and $q(t)$ is defined on the set of integers $[a+1, b+1]$.

Equation (2.38) may be written in the more familiar form

$$p(t)y(t+1) + c(t)y(t) + p(t-1)y(t-1) = 0, \tag{2.39}$$

where

$$c(t) = q(t) - p(t) - p(t-1) \text{ for } t \in [a+1, b+1]. \tag{2.40}$$

Remark 2.14. *Any equation written in the form of equation (2.39), where $p(t) > 0$ on $[a+1, b+1]$, can be written in the self-adjoint form of equation (2.38) by taking*

$$q(t) = c(t) + p(t) + p(t-1). \tag{2.41}$$

In fact, any equation of the form

$$\alpha(t)y(t+1) + \beta(t)y(t) + \gamma(t)y(t-1) = 0, \tag{2.42}$$

where $\alpha(t) > 0$ on $[a, b+1]$, $\gamma(t) > 0$ on $[a+1, b+1]$, can be written in the self-adjoint form of the equation (2.38) or (2.39). To find $p(t)$ and $q(t)$ from $\alpha(t), \beta(t)$ and $\gamma(t)$, we multiply both sides of the equation (2.42) by a positive function $h(t)$ to obtain

$$\alpha(t)h(t)y(t+1) + \beta(t)h(t)y(t) + \gamma(t)h(t)y(t-1) = 0. \tag{2.43}$$

Comparing (2.43) with (2.39), we obtain

$$\alpha(t)h(t) = p(t),$$

$$\gamma(t)h(t) = p(t-1).$$

Thus

$$\alpha(t)h(t) = \gamma(t+1)h(t+1),$$

or

$$h(t+1) = \frac{\alpha(t)}{\gamma(t+1)}h(t), \quad t \in [a, b]. \tag{2.44}$$

Hence

$$h(t) = A\prod_{s=a}^{t-1} \frac{\alpha(s)}{\gamma(s+1)},$$

where A is any positive constant. This gives us

$$p(t) = A\alpha(t) \prod_{s=a}^{t-1} \frac{\alpha(s)}{\gamma(s+1)} \cdot$$

Also, from (2.41) we obtain $q(t) = \beta(t)h(t) + p(t) + p(t-1)$, then we have that equation (2.42) is equivalent to (2.38).

Example 2.8. *Write the following equation in the self-adjoint form,*

$$2^t y(t+1) + (\sin t - 3.2^{t-1})y(t) + 2^{t-1}y(t-1) = 0 \cdot$$

Here $p(t) = 2^t$ *and* $c(t) = \sin t - 3.2^{t-1}$. *Hence,*

$$q(t) = c(t) + p(t) + p(t-1) = \sin t - 3.2^{t-1} + 2^t + 2^{t-1} = \sin t.$$

Then the self-adjoint form of this equation is

$$\Delta(2^{t-1}\Delta y(t-1)) + \sin t y(t) = 0.$$

In what follows, we define a linear operator L on the set $\{y : [a, b+2] \to \mathbb{R}\}$ by:

$$Ly(t) = \Delta(p(t-1)\Delta y(t-1)) + q(t)y(t), \text{ for } t \in [a+1, b+1]. \tag{2.45}$$

Definition 2.14. *The Cauchy function* $z = z(t,s)$, *defined for* $a \le t \le b+2$ *and* $a+1 \le s \le b+1$, *is defined as the function that, for each fixed* s *in* $[a+1, b+1]$, *is the solution of the initial value problem*

$$\begin{aligned} Lz(t,s) &= 0, \\ z(s,s) &= 0, \\ z(s+1,s) &= \tfrac{1}{p(s)} \cdot \end{aligned}$$

Example 2.9. *For* s *fixed in* $[a+1, b+1]$, *find the Cauchy function for*

$$\Delta(p(t-1)\Delta y(t-1)) = 0, \quad t \ge s.$$

Since for each fixed s *the Cauchy function for this difference equation is a solution for*

$$\Delta(p(t-1)\Delta z(t-1,s)) = 0, \quad t \in [a+1, b+1].$$

Therefore there is a constant $\alpha(s)$ *such that*

$$p(t-1)\Delta z(t-1,s) = \alpha(s), \quad t \in [a+1, b+2].$$

From the initial conditions, for $t = s+1$, *we find that* $\alpha(s) = 1$.

So replacing t *by* $t+1$ *yields*

$$\Delta z(t,s) = \frac{1}{p(t)}.$$

Assuming that $t \geq s$ *and summing from* s *to* $t-1$, *we obtain*

$$z(t,s) - z(s,s) = \sum_{\tau=s}^{t-1} \frac{1}{p(\tau)}.$$

Then the Cauchy function is

$$z(t,s) = \sum_{\tau=s}^{t-1} \frac{1}{p(\tau)}, \quad t \geq s.$$

Example 2.10. *The difference equation* $\Delta^2 y(t-1) = 0$ *has the Cauchy function*

$$z(t,s) = t-s, \ t \geq s.$$

Theorem 2.11. *[56, Theorem 6.3, page 236] If* u_1, u_2 *are two linearly independent solutions of equation (2.38), then the Cauchy function for (2.38) is given by:*

$$z(t,s) = \frac{\begin{vmatrix} u_1(s) & u_2(s) \\ u_1(t) & u_2(t) \end{vmatrix}}{p(s)\begin{vmatrix} u_1(s) & u_2(s) \\ u_1(s+1) & u_2(s+1) \end{vmatrix}},$$

for $a \leq t \leq b+2, a+1 \leq s \leq b+1$.

Example 2.11. *Find the Cauchy function of the following difference equation using Theorem 2.11.*

$$\Delta(p(t-1)\Delta y(t-1)) = 0, \quad t \geq s.$$

We have $u_1(t) = 1$ *et* $u_2(t) = \sum\limits_{\tau=a}^{t-1} \frac{1}{p(\tau)}$ *are two linearly independent solutions of the equation.*

Then, from Theorem 2.11, we have

$$z(t,s) = \frac{\begin{vmatrix} 1 & \sum\limits_{\tau=a}^{s-1} \frac{1}{p(\tau)} \\ 1 & \sum\limits_{\tau=a}^{t-1} \frac{1}{p(\tau)} \end{vmatrix}}{p(s)\begin{vmatrix} 1 & \sum\limits_{\tau=a}^{s-1} \frac{1}{p(\tau)} \\ 1 & \sum\limits_{\tau=a}^{s} \frac{1}{p(\tau)} \end{vmatrix}} = \sum_{\tau=s}^{t-1} \frac{1}{p(\tau)}.$$

Linear self-adjoint equations with initial conditions

The following two results show how the Cauchy function is used to solve an initial non-homogeneous problem.

Theorem 2.12. *The solution of the initial value problem*

$$\begin{aligned} Ly(t) &= h(t), \quad t \in [a+1, b+1] \\ y(a) &= 0, \\ y(a+1) &= 0, \end{aligned}$$

is given by

$$y(t) = \sum_{s=a+1}^{t} z(t,s)h(s), \tag{2.46}$$

for $t \in [a, b+2]$, where z is the Cauchy function for $Ly(t) = 0$. (Here, if $t = b+2$, then the term $z(b+2, b+2)h(b+2)$ is understood to be zero.)

Proof. Let the function y be given by (2.46). By convention $y(a) = 0$. Also,

$$\begin{aligned} y(a+1) &= z(a+1, a+1)h(a+1) = 0, \\ y(a+2) &= z(a+2, a+1)h(a+1) + z(a+2, a+2)h(a+2) \\ &= \tfrac{h(a+1)}{p(a+1)}. \end{aligned}$$

Then y satisfies $Ly(t) = h(t)$ for $t = a+1$.

Assume that $a+2 \le t \le b+1$. Then

$$\begin{aligned} Ly(t) &= p(t-1)y(t-1) + c(t)y(t) + p(t)y(t+1) \\ &= \sum_{s=a+1}^{t-1} p(t-1)z(t-1,s)h(s) + \sum_{s=a+1}^{t} c(t)z(t,s)h(s) + \sum_{s=a+1}^{t+1} p(t)z(t+1,s)h(s) \\ &= \sum_{s=a+1}^{t-1} Lz(t,s)h(s) + c(t)z(t,t)h(t) + p(t)z(t+1,t)h(t) + p(t)z(t+1,t+1)h(t+1) \\ &= h(t). \end{aligned}$$

□

Remark 2.15. *1. The formula (2.46) is called variation of constants formula.*

2. In the variation of constants formula, we only need to know the Cauchy function for $t \geq s$.

Corollary 2.6. *The solution of the non-homogenous initial value problem*

$$\begin{aligned} Ly(t) &= h(t), \quad t \in [a+1, b+1] \\ y(a) &= A, \\ y(a+1) &= B, \end{aligned}$$

is given by

$$y(t) = u(t) + \sum_{s=a+1}^{t} z(t,s)h(s), \tag{2.47}$$

where z is the Cauchy function for $Ly(t) = 0$ and u is the solution of the initial value problem $Lu(t) = 0$, $u(a) = A$, $u(a+1) = B$.

Proof. Since u is the solution of $Lu(t) = 0$ and $\sum_{s=a+1}^{t} z(t,s)h(s)$ is a solution of $Ly(t) = h(t)$, then

$$y(t) = u(t) + \sum_{s=a+1}^{t} z(t,s)h(s)$$

is a solution of $Ly(t) = h(t)$. Also, $y(a) = u(a) = A$ and $y(a+1) = u(a+1) = B$. □

Linear self-adjoint equations with boundary conditions

In [50], Hartman introduced the notion of generalized zeros in order to obtain a discrete analogue of Sturm's separation theorem in differential equations. This concept provides a mechanism for obtaining fundamental results about second order self-adjoint equations and also represents the best approach for extending these results to higher order equations. The concept of disconjugacy of difference equation will be introduced in this section, and we will see its importance in obtaining an existence and uniqueness result for solutions of boundary value problems and the existence and uniqueness of the Green's functions.

The following lemma shows that there is no nontrivial solution of equation (2.38) with $y(t_0) = 0$ and $y(t_0 - 1)y(t_0 + 1) \geq 0, t_0 > a$. In some sense this lemma says that nontrivial solutions of equation (2.38) can have only simple zeros.

Lemma 2.4. *If y is a nontrivial solution of equation (2.38) such that $y(t_0) = 0, a < t_0 < b+2$, then $y(t_0 - 1)y(t_0 + 1) < 0$.*

Proof. Since $y(t)$ is a nontrivial solution of equation (2.38) with $y(t_0) = 0, a < t_0 < b + 2$, we obtain from equation (2.39)

$$p(t_0)y(t_0 + 1) = -p(t_0 - 1)y(t_0 - 1),$$

$$\frac{y(t_0 + 1)}{y(t_0 - 1)} = -\frac{p(t_0 - 1)}{p(t_0)} < 0.$$

Since $p(t) > 0$, then $y(t_0 + 1)$, $y(t_0 - 1) \neq 0$, it follows that

$$y(t_0 - 1)y(t_0 + 1) < 0.$$

□

Definition 2.15. *A solution y of the equation (2.38) has a generalized zero at t_0 provided that $y(t_0) = 0$ if $t_0 = a$ and if $t_0 > a$ either $y(t_0) = 0$ or $y(t_0 - 1)y(t_0) < 0$.*
In other words, a generalized zero of a solution is either an actual zero or where the solution changes its sign.

Theorem 2.13. *(Sturm separation theorem)[35, Theorem 7.9, page 321]*
Let y_1 and y_2 be two linearly independent solutions of (2.38). Then the following statements hold:

(i) y_1 and y_2 cannot have a common zero, that is, if $y_1(t_0) = 0$, then $y_2(t_0) \neq 0$.

(ii) If y_1 has a zero at t_1 and a generalized zero at $t_2 > t_1$, then y_2 must have a generalized zero in $(t_1, t_2]$.

(iii) If y_1 has a generalized zero at t_1 and $t_2 > t_1$, then y_2 must have a generalized zero in $[t_1, t_2]$.

Definition 2.16. *We say that the difference equation (2.38) is "disconjugate" on $[a, b+2]$ provided that no nontrivial solution of (2.38) has two or more generalized zeros on $[a, b+2]$. Of course, in any interval $[a, b+2]$ there is a nontrivial solution with at least one generalized zero.*

Example 2.12. *The difference equation*

$$y(t+1) - \sqrt{3}y(t) + y(t-1) = 0,$$

is disconjugate on any interval of length less than 6. This follows from the fact that any solution of this equation is of the form: $y(t) = c_1 \sin(\frac{\pi t}{6} + c_2), \quad c_1, c_2 \in \mathbb{R}$.
However, if we consider the difference equation

$$y(t+2) - 7y(t+1) + 12y(t) = 0,$$

we find that it is disconjugate on any interval because its solution is given by: $y(t) = c_1 4^t + c_2 3^t$, with $c_1, c_2 \in \mathbb{R}$.

Theorem 2.14. *The difference equation $Ly(t) = 0$ is disconjugate on $[a, b+2]$ if and only if there is a positive solution of $Ly(t) = 0$ on $[a, b+2]$.*

Proof. Assume that $Ly(t) = 0$ is disconjugate on $[a, b+2]$. Let $u(t), v(t)$ be solutions of $Ly(t) = 0$, satisfying

$$u(a) = 0, \quad u(a+1) = 1,$$

$$v(b+1) = 1, \quad v(b+2) = 0.$$

By the disconjugacy, $u(t) > 0$ on $[a+1, b+2]$ and $v(t) > 0$ on $[a, b+1]$. It follows that $y(t) = u(t) + v(t)$ is a positive solution of $Ly(t) = 0$.
Conversely assume that $Ly(t) = 0$ has a positive solution on $[a, b+2]$. It follows from the Sturm separation theorem that no nontrivial solution has two generalized zeros in $[a, b+2]$. □

Consider the boundary value problem :

$$\Delta^2 y(t-1) + 2y(t) = 0,$$

$$y(0) = A, \quad y(2) = B.$$

- If $A = B = 0$, this boundary value problem has infinitely many solutions.
- If $A = 0,\ B \neq 0$, it has no solutions.

In the following theorem we show that with the assumption of disconjugacy this type of boundary value problem has a unique solution.

Theorem 2.15. *[56, Theorem 6.7, page 243] If the equation $Ly(t) = 0$ is disconjugate on $[a, b+2]$, then the boundary value problem*

$$\begin{aligned} Ly(t) &= h(t), \\ y(t_1) &= A, \\ y(t_2) &= B, \end{aligned}$$

where $a \leq t_1 < t_2 \leq b+2,\ A, B \in \mathbb{R}$, has a unique solution.

Proof. Let y_1, y_2 be two linearly independent solutions of $Ly(t) = 0$ and let y_p be a particular solution of $Ly(t) = h(t)$, then a general solution of $Ly(t) = h(t)$ is

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t), \quad C_1, C_2 \in \mathbb{R}\cdot$$

The boundary conditions lead to the system of equations

$$C_1 y_1(t_1) + C_2 y_2(t_1) = A - y_p(t_1),$$

$$C_1 y_1(t_2) + C_2 y_2(t_2) = B - y_p(t_2).$$

This system has a unique solution if and only if

$$\begin{vmatrix} y_1(t_1) & y_2(t_1) \\ y_1(t_2) & y_2(t_2) \end{vmatrix} \neq 0.$$

Assume that

$$\begin{vmatrix} y_1(t_1) & y_2(t_1) \\ y_1(t_2) & y_2(t_2) \end{vmatrix} = 0,$$

Then there are constants d_1, d_2 ($d_1 = y_2(t_2)$ or $d_1 = -y_2(t_1)$ and $d_2 = -y_1(t_2)$ or $d_2 = y_1(t_1)$), not both zero, such that the nontrivial solution

$$y(t) = d_1 y_1(t) + d_2 y_2(t)$$

satisfies

$$y(t_1) = y(t_2) = 0.$$

This contradicts the disconjugacy of $Ly(t) = 0$ on $[a, b+2]$. □

2.6 Green's functions

2.6.1 Green's function for conjugate boundary value problems

In this section we introduce the Green's function for a two point conjugate boundary value problem. It will follow that under certain conditions the solution of a non-homogeneous BVP can be expressed in terms of Green's functions. By Theorem 2.15, If $Ly(t) = 0$ is disconjugate on $[a, b+2]$, then the BVP

$$Ly(t) = h(t), \quad t \in [a+1, b+1] \tag{2.48}$$

$$y(a) = 0, \tag{2.49}$$

$$y(b+2) = 0, \tag{2.50}$$

has a unique solution y. We would like to have a formula like the variation of constants formula for this solution. First, assume that there is a function $G = G(t,s)$ that satisfies the following:

(a) $G(t,s)$ is defined for $a \le t \le b+2, a+1 \le s \le b+1$.

(b) $LG(t,s) = \delta_{ts}$ for $a+1 \le t \le b+1, a+1 \le s \le b+1$, where δ_{ts} is the Kronecker delta ($\delta_{ts} = 0$ if $t \neq s, \delta_{ts} = 1$ if $t = s$).

(c) $G(a,s) = G(b+2,s) = 0$, $a+1 \le s \le b+1$.

We set

$$y(t) = \sum_{s=a+1}^{b+1} G(t,s)h(s), \quad t \in [a, b+2].$$

We claim that y satisfies (2.48)-(2.50). First by (c), we have

$$y(a) = \sum_{s=a+1}^{b+1} G(a,s)h(s) = 0$$

and

$$y(b+2) = \sum_{s=a+1}^{b+1} G(b+2,s)h(s) = 0,$$

then (2.49) and (2.50) hold. Next, for $a+1 \leq t \leq b+1$, we have

$$\begin{aligned} Ly(t) &= \sum_{s=a+1}^{b+1} LG(t,s)h(s) \\ &= \sum_{s=a+1}^{b+1} \delta_{ts}h(s) \\ &= h(t). \end{aligned}$$

Thus we have shown that if there is a function $G = G(t,s)$ satisfying $(a)-(c)$, then the function y defined by $y(t) = \sum\limits_{s=a+1}^{b+1} G(t,s)h(s)$ satisfies the BVP (2.48)-(2.50).

We show that if $Ly(t) = 0$ is disconjugate on $[a, b+2]$, then there is a function G satisfying $(a)-(c)$, where the operator L is given by (2.45).

Let y_1 be the solution to the initial value problem $Ly(t) = 0,\ t \in [a, b+2]$, $y_1(a) = 0, y_1(a+1) = 1$, and let $z = z(t,s)$ be the Cauchy function for $Ly(t) = 0$.

For $a \leq t \leq b+2,\ a+1 \leq s \leq b+1$ define $G(t,s)$ by:

$$G(t,s) = \begin{cases} -\frac{z(b+2,s)}{y_1(b+2)} y_1(t), & t \leq s \\ z(t,s) - \frac{z(b+2,s)}{y_1(b+2)} y_1(t), & s \leq t. \end{cases} \tag{2.51}$$

Note that

- Since $Ly(t) = 0$ is disconjugate on $[a, b+2]$, so $y_1(b+2) > 0$, we are not dividing by zero in the definition of $G(t,s)$.
- Since $z(s,s) = 0$, we may write $t \leq s$ and $s \leq t$ in the definition of $G(t,s)$.

We have

$$G(a,s) = -\frac{z(b+2,s)}{y_1(b+2)} y_1(a) = 0$$

and

$$G(b+2,s) = z(b+2,s) - \frac{z(b+2,s)}{y_1(b+2)} y_1(b+2) = 0,$$

then $G(t,s)$ satisfies (c).

Next we show that G satisfies (b). We distinguish three cases:

- If $t \geq s+1$, then

$$LG(t,s) = Lz(t,s) - \frac{z(b+2,s)}{y_1(b+2)} Ly_1(t) = 0.$$

- If $t \leq s-1$, then

$$LG(t,s) = -\frac{z(b+2,s)}{y_1(b+2)} Ly_1(t) = 0.$$

- If $t = s$,

$$\begin{aligned} LG(s,s) &= p(s)G(s+1,s) + c(s)G(s,s) + p(s-1)G(s-1,s) \\ &= p(s)z(s+1,s) - \frac{z(b+2,s)}{y_1(b+2)} Ly_1(s) \\ &= 1. \end{aligned}$$

Hence G satisfies $(a)-(c)$.

Next, we show that if $Ly(t) = 0$ is disconjugate on $[a, b+2]$, there is a unique function satisfying $(a)-(c)$.

We know that G defined by equation (2.51) satisfies $(a)-(c)$. Assume that $H = H(t,s)$ satisfies $(a)-(c)$. Fix $s \in [a+1, b+1]$ and set

$$z(t) = G(t,s) - H(t,s).$$

It follows from (b) that z is a solution of $Ly(t) = 0,\ t \in [a, b+2]$.

By (c), we obtain $z(a) = 0,\ z(b+2) = 0$. Since $Ly(t) = 0$ is disconjugate on $[a, b+2]$, we must have $z \equiv 0$ on $[a, b+2]$. Since $s \in [a+1, b+1]$ is arbitrary, it follows that

$$G(t,s) = H(t,s), \qquad \text{for } a \leq t \leq b+2,\ a+1 \leq s \leq b+1.$$

Definition 2.17. *If $Ly(t) = 0$ is disconjugate on $[a, b+2]$, we define the Green's function for the boundary value problem $Ly(t) = 0,\ y(a) = 0,\ y(b+2) = 0$ to be the unique function $G(t,s)$ satisfying $(a)-(c)$.*

Theorem 2.16. *[56, Theorem 6.8, page 246] If the equation $Ly(t) = 0$ is disconjugate on $[a, b+2]$, then the boundary value problem*

$$Ly(t) = h(t), \quad t \in [a+1, b+1]$$

$$y(a) = 0 = y(b+2),$$

has a unique solution given by

$$y(t) = \sum_{s=a+1}^{b+1} G(t,s)h(s), \quad t \in [a, b+2],$$

where G is the Green's function defined by:

$$G(,s) = \begin{cases} -\frac{z(b+2,s)}{y_1(b+2)} y_1(t), & t \leq s \\ z(t,s) - \frac{z(b+2,s)}{y_1(b+2)} y_1(t), & t \geq s. \end{cases}$$

Moreover, G verify $G(t,s) < 0$ on the square $a+1 \leq t, s \leq b+1$.

Proof. It remains to show that $G(t,s) < 0$ on the square $a+1 \leq t, s \leq b+1$.

To see this, fix $s \in [a+1, b+1]$. Since $Ly(t) = 0$ is disconjugate on $[a, b+2]$, then $y_1(t) > 0$ for $a < t \leq b+2$ and $z(t,s) > 0$ for $s < t \leq b+2$. Hence

$$G(t,s) = -\frac{z(b+2,s)}{y_1(b+2)} y_1(t) < 0, \quad a+1 \leq t \leq s,$$

and

$$G(t,s) = z(t,s) - \frac{z(b+2,s)}{y_1(b+2)} y_1(t), \quad s \leq t \leq b+2,$$

which as a function of t is a solution of $Ly(t) = 0$ on $[a, b+2]$. Since $G(b+2, s) = 0$ and $G(s,s) < 0$, we have that

$$G(t,s) < 0, \quad s \leq t \leq b+1.$$

Since $s \in [a+1, b+1]$ is arbitrary, we get the desired result. □

Example 2.13. *Consider the boundary value problem*

$$\Delta(p(t-1)\Delta y(t-1)) = 0,$$

$$y(a) = y(b+2) = 0\cdot$$

From example 2.9 the Cauchy function is given by

$$z(t,s) = \sum_{\tau=s}^{t-1} \frac{1}{p(\tau)}, \quad t \geq s.$$

The solution that satisfies the initial conditions $y(a) = 0,\ y(a+1) = 1,$ *is*

$$y_1(t) = z(t,a) = \sum_{\tau=a}^{t-1} \frac{1}{p(\tau)}$$

Then,

For $t \leq s$

$$\begin{aligned} G(t,s) &= -\frac{z(b+2,s)}{y_1(b+2)} y_1(t) \\ &= -\frac{\sum\limits_{\tau=s}^{b+1} \frac{1}{p(\tau)}}{\sum\limits_{\tau=a}^{b+1} \frac{1}{p(\tau)}} \sum_{\tau=a}^{t-1} \frac{1}{p(\tau)}. \end{aligned}$$

For $t \geq s$

$$\begin{aligned} G(t,s) &= z(t,s) - \frac{z(b+2,s)}{y_1(b+2)} y_1(t) \\ &= \sum_{\tau=s}^{t-1} \frac{1}{p(\tau)} - \frac{\sum\limits_{\tau=s}^{b+1} \frac{1}{p(\tau)}}{\sum\limits_{\tau=a}^{b+1} \frac{1}{p(\tau)}} \sum_{\tau=a}^{t-1} \frac{1}{p(\tau)} \\ &= -\frac{\sum\limits_{\tau=a}^{s-1} \frac{1}{p(\tau)}}{\sum\limits_{\tau=a}^{b+1} \frac{1}{p(\tau)}} \sum_{\tau=t}^{b+1} \frac{1}{p(\tau)}. \end{aligned}$$

Then the Green function is given by

$$G(t,s) = \begin{cases} -\frac{\sum\limits_{\tau=s}^{b+1} \frac{1}{p(\tau)}}{\sum\limits_{\tau=a}^{b+1} \frac{1}{p(\tau)}} \sum\limits_{\tau=a}^{t-1} \frac{1}{p(\tau)}, & \text{if } t \leq s \\ \\ -\frac{\sum\limits_{\tau=a}^{s-1} \frac{1}{p(\tau)}}{\sum\limits_{\tau=a}^{b+1} \frac{1}{p(\tau)}} \sum\limits_{\tau=t}^{b+1} \frac{1}{p(\tau)}, & \text{if } s \leq t. \end{cases}$$

In particular, the Green function to the problem

$$\Delta^2 y(t-1) = 0,$$

$$y(a) = y(b+2) = 0$$

is defined by:

$$G(t,s) = \begin{cases} -\frac{(t-a)(b+2-s)}{b+2-a}, & si\ \ t \leq s \\ -\frac{(s-a)(b+2-t)}{b+2-a}, & si\ \ s \leq t. \end{cases} \tag{2.52}$$

Corollary 2.7. *[56, Corollary 6.4, page 249] If $Ly(t) = 0$ is disconjugate on $[a, b+2]$, the unique solution of the boundary value problem*

$$Ly(t) = h(t),$$

$$y(a) = A,\ y(b+2) = B,$$

is given by:

$$y(t) = u(t) + \sum_{s=a+1}^{b+1} G(t,s)h(s),$$

where G is the Green's function for the BVP $Ly(t) = 0$, $y(a) = 0 = y(b+2)$ and u is the solution of the BVP $Lu(t) = 0$, $u(a) = A$, $u(b+2) = B$.

Example 2.14. *Consider the following boundary value problem:*

$$\Delta^2 y(t-1) = 12, \quad t \in [1,5]$$

$$y(0) = 1,\ y(6) = 7.$$

By Theorem 2.16 , the solution to the associated homogenous problem is given by:

$$y(t) = \sum_{s=1}^{s=5} 12\, G(t,s),$$

where, from Example 2.13,

$$G(t,s) = \begin{cases} -\frac{(6-s)t}{6}, & t \le s \\ -\frac{(6-t)s}{6}, & s \le t. \end{cases}$$

Hence the solution to the homogenous problem is

$$y(t) = 6t^2 - 36t, \quad t \in [0,6].$$

From Corollary 2.7, the general solution is given by

$$y(t) = u(t) + 6t^2 - 36t, \quad t \in [0,6],$$

where u is a solution to the problem :

$$\Delta^2 u(t-1) = 0,$$

$$u(0) = 1,\ u(6) = 7.$$

We find that $u(t) = 1 + t$. Then, $y(t) = 6t^2 - 35t + 1, \quad t \in [0,6]$.

Definition 2.18. *We say that the difference equation $Ly(t) = 0$ is "disfocal" on $[a, b+2]$ if there is no nontrivial solution y such that y has a generalized zero at t_1 and Δy has a generalized zero at t_2, where $a \leq t_1 \leq t_2 \leq b+1$.*

Theorem 2.17. *[56] If $Ly(t) = 0$ is disfocal on $[a, b+2]$, then the BVP*

$$\begin{aligned} Ly(t) &= h(t), \\ y(t_1) &= A, \\ \Delta y(t_2) &= B, \end{aligned}$$

where $a \leq t_1 < t_2 \leq b+1, A, B$ constants, has a unique solution.

Proof. Let y_1, y_2 be two linearly independent solutions of $Ly(t) = 0$ and let y_p be a particular solution of $Ly(t) = h(t)$, then a general solution of $Ly(t) = h(t)$ is

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t), \quad C_1, C_2 \in \mathbb{R}\cdot$$

The boundary conditions lead to the system of equations

$$C_1 y_1(t_1) + C_2 y_2(t_1) = A - y_p(t_1),$$

$$C_1 \Delta y_1(t_2) + C_2 \Delta y_2(t_2) = B - \Delta y_p(t_2).$$

This system has a unique solution if and only if

$$\begin{vmatrix} y_1(t_1) & y_2(t_1) \\ \Delta y_1(t_2) & \Delta y_2(t_2) \end{vmatrix} \neq 0.$$

Assume that

$$\begin{vmatrix} y_1(t_1) & y_2(t_1) \\ \Delta y_1(t_2) & \Delta y_2(t_2) \end{vmatrix} = 0,$$

Then there are constants d_1, d_2 ($d_1 = \Delta y_2(t_2)$ or $d_1 = -y_2(t_1)$ and $d_2 = -\Delta y_1(t_2)$ or $d_2 = y_1(t_1)$) not both zero, such that the nontrivial solution

$$y(t) = d_1 y_1(t) + d_2 y_2(t),$$

with

$$\Delta y(t) = d_1 \Delta y_1(t) + d_2 \Delta y_2(t)$$

satisfies

$$y(t_1) = \Delta y(t_2) = 0.$$

This contradicts the disfocality of $Ly(t) = 0$ on $[a, b+2]$. □

2.6.2 Green's function for focal boundary value problems

In this section we introduce the Green's function for a focal boundary value problem. It will follow that under certain conditions the solution of a non-homogeneous BVP can be expressed in terms of Green's functions. By Theorem 2.17, If $Ly(t) = 0$ is disfocal on $[a, b+2]$, then the BVP

$$Ly(t) = h(t), \quad t \in [a+1, b+1] \tag{2.53}$$

$$y(a) = 0, \tag{2.54}$$

$$\Delta y(b+1) = 0, \tag{2.55}$$

has a unique solution y.

First, assume that there is a function $G = G(t, s)$ that satisfies these properties:

(a) $G(t, s)$ is defined for $a \le t \le b+2, a+1 \le s \le b+1$.

(b) $LG(t, s) = \delta_{ts}$ for $a \le t \le b+2, a+1 \le s \le b+1$.

(c) $G(a, s) = \Delta G(b+1, s) = 0$, $a+1 \le s \le b+1$.

We set

$$y(t) = \sum_{s=a+1}^{b+1} G(t, s)h(s), \quad t \in [a, b+2].$$

We claim that y satisfies (2.53)-(2.55). First by (c), we have

$$y(a) = \sum_{s=a+1}^{b+1} G(a, s)h(s) = 0$$

and

$$\Delta y(b+1) = \sum_{s=a+1}^{b+1} \Delta G(b+1, s)h(s) = 0.$$

Then the boundary conditions (2.54) and (2.55) hold.

Next, for $a+1 \le t \le b+1$ we obtain

$$\begin{aligned} Ly(t) &= \sum_{s=a+1}^{b+1} LG(t,s)h(s) \\ &= \sum_{s=a+1}^{b+1} \delta_{ts}h(s) \\ &= h(t). \end{aligned}$$

Thus we have shown that if there is a function $G = G(t,s)$ satisfying $(a)-(c)$, then $y(t) = \sum\limits_{s=a+1}^{b+1} G(t,s)h(s)$ is a solution to the problem (2.53)- (2.55).

Know, we show that if $Ly(t) = 0$ is disfocal on $[a, b+2]$, then there is a function G satisfying the conditions $(a)-(c)$. Let $y_1(t)$ be the solution to the initial value problem (2.38), $y_1(a) = 0, y_1(a+1) = 1$, and let $z(t,s)$ be the Cauchy function for $Ly(t) = 0$.

Since $Ly(t) = 0$ is disfocal on $[a, b+2]$, then y_1 satisfies $\Delta y_1(t) \neq 0$ for $a+1 \le t \le b+1$. Define $G(t,s)$ for $a \le t \le b+2, a+1 \le s \le b+1$ by:

$$G(t,s) = \begin{cases} -\frac{\Delta y(b+1,s)}{\Delta y_1(b+1)} y_1(t), & t \le s \\ y(t,s) - \frac{\Delta y(b+1,s)}{\Delta y_1(b+1)} y_1(t), & s \le t. \end{cases} \tag{2.56}$$

We have

$$G(a,s) = -\frac{\Delta y(b+1,s)}{\Delta y_1(b+1)} y_1(a) = 0$$

and

$$\Delta G(b+1,s) = \Delta y(b+1,s) - \frac{\Delta y(b+1,s)}{\Delta y_1(b+1)} y_1(b+1) = 0,$$

then $G(t,s)$ satisfies (c).

Next we show that G satisfies (b). We distinguish three cases:

- If $t \ge s+1$, then

$$LG(t,s) = Ly(t,s) - \frac{\Delta y(b+1,s)}{\Delta y_1(b+1)} Ly_1(t) = 0.$$

- If $t \le s-1$, then

$$LG(t,s) = -\frac{\Delta y(b+1,s)}{\Delta y_1(b+1)} Ly_1(t) = 0.$$

- If $t = s$, then

$$\begin{aligned} LG(s,s) &= p(s)G(s+1,s) + c(s)G(s,s) + p(s-1)G(s-1,s) \\ &= p(s)y(s+1,s) - \frac{\Delta y(b+1,s)}{\Delta y_1(b+1)} Ly_1(s) \\ &= 1. \end{aligned}$$

Hence $G(t,s)$ satisfies $(a) - (c)$.

Next, we show that if $Ly(t) = 0$ is disfocal on $[a, b+2]$, there is a unique function G satisfying $(a) - (c)$. We know that $G(t,s)$ defined by equation (2.56) satisfies $(a) - (c)$. Assume that $H(t,s)$ satisfies $(a) - (c)$. Fix $s \in [a+1, b+1]$ and set

$$z(t) = G(t,s) - H(t,s).$$

It follows from (b) that z is a solution of $Ly(t) = 0$ on $[a, b+2]$.

By (c), $z(a) = 0$, $\Delta z(b+2) = 0$. Since $Ly(t) = 0$ is disfocal on $[a, b+2]$, we must have $z(t) = 0$ on $[a, b+2]$. Since $s \in [a+1, b+1]$ is arbitrary, it follows that

$$G(t,s) = H(t,s), \qquad \text{for } a \le t \le b+2,\ a+1 \le s \le b+1.$$

Definition 2.19. *If $Ly(t) = 0$ is disfocal on $[a, b+2]$, then there is a unique function $G(t,s)$ satisfying properties $(a) - (c)$. This function $G(t,s)$ is called the Green's function for the boundary value problem $Ly(t) = 0$, $t \in [a+1, b+1]$, $y(a) = \Delta y(b+1) = 0$.*

Theorem 2.18. *[56] If $Ly(t) = 0$ is disfocal on $[a, b+2]$, then the unique solution of*

$$Ly(t) = h(t),$$

$$y(a) = \Delta y(b+1) = 0,$$

is given by

$$y(t) = \sum_{s=a+1}^{b+1} G(t,s)h(s),$$

where,

$$G(t,s) = \begin{cases} -\frac{\Delta y(b+1,s)}{\Delta y_1(b+1)} y_1(t), & t \le s \\ y(t,s) - \frac{\Delta y(b+1,s)}{\Delta y_1(b+1)} y_1(t), & t \ge s. \end{cases}$$

Furthermore, $G(t,s) \le 0$ on $a \le t \le b+1$, $a+1 \le s \le b+1$.

3 Nonlinear first order impulsive difference equations

The results of this chapter are obtained by Bouchal, Mebarki and Georgiev in [24].

3.1 Introduction

This chapter is devoted to investigate the following boundary value problem for impulsive difference equations with nonlinear two point functional boundary conditions:

$$\begin{aligned} \Delta x(n) &= f(n, x(n)), \quad n \neq n_k, \quad n \in J, \\ \Delta x(n_k) &= I_k(x(n_k)), \quad n = n_k, \\ Mx(0) - Nx(T) &= g(x(0), x(T)), \end{aligned} \tag{3.1}$$

where Δ is the forward difference operator, i.e., $\Delta x(n) = x(n+1) - x(n)$, $J = [0, T] \cap \mathbb{N}$, $T \in \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers, $M, N > 0$, $f \in \mathcal{C}(J \times \mathbb{R})$, $g \in \mathcal{C}(\mathbb{R} \times \mathbb{R})$, $I_k \in \mathcal{C}(\mathbb{R})$, $k \in \{1, \ldots, p\}$, $\{n_k\}_{k=1}^{p}$ are fixed impulsive points such that

$$0 < n_1 < n_2 < \ldots < n_p < T, \quad p \in \mathbb{N}.$$

Differential equations and difference equations serve as descriptions for many real world phenomena that are explored in applied sciences. However, many of them, including models for neural networks, populations dynamics, optimal control, control theory, economics, industrial robotics, and medicines, etc, experience abrupt changes in their states whose duration is negligible in comparison with the duration of the process, often these short term disruptions are represented in the form of impulses treated mathematically by an impulsive equations. For further information on impulsive equations, we refer the reader to the books [62, 1].

There are three components that compose an impulsive difference equation:

- a difference equation, which describes the state of the system between impulses.
- an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs.
- a jump criterion, which defines a set of jump events in which the impulse equation appears.

3.2 Historical notes and motivations

In order to represent the development of a real process with a short-term perturbation, it is sometimes convenient to treat these perturbations as "instantaneous" which is described mathematically by impulsive equations. Such problems attracted the attention of physicists, because they was aware that the modelisation of many applied problems arising in several sciences and engineering fields are pointless without the dependence with impulses states, and these equations gave the possibility to adequately describe a variety of nonlinear phenomena. In 1960, Mil'man and Myshkis introduced the work on impulsive ordinary differential equations in their paper entitled "On the stability of motion in the presence of impulses" [69], where they gave some general concepts about the systems with impulses and they obtained the first result on stability of solutions of such systems.

In the literature, there exist a great number of works devoted to the study of first order impulsive differential equations with different types of boundary conditions such as: periodic [52, 51, 37, 83], anti-periodic [32, 66, 38], multi-point, nonlinear, nonlocal and integral boundary conditions [28, 92, 8] and references therein. In comparison to their discrete analogues, there are fewer works devoted to the study of first order discrete impulsive equations. The well-known methods used to deal with first order impulsive difference equations are the method of upper and lower solutions, the monotone iterative technique and fixed point theory. In the following, we summarize some works:

When the function g in the BVP (3.1) is a constant, criteria on the existence of minimal and maximal solutions to this problem are obtained in [85] by using a comparison theorem and the method of upper and lower solutions coupled with the monotone iterative technique. Note that, the principle idea of this method is that by making use of the upper and lower solution as an initial iteration one can construct monotone sequences from the corresponding linear equation, and then these sequences converge monotonically to the maximal and minimal solution of the nonlinear equation.

In 2018, Tian et al. [82] investigated periodic boundary value problems for first order impulsive difference equations with time delay, by utilizing the combination of these two methods where an existence theorem of extremal solutions is obtained. The authors in [55, 84] analyzed by using the same approach the existence of solutions for a first order functional difference equations without impulse effects with nonlinear functional boundary conditions of the form $g(x(0), x(T)) = 0$ and $g(x(0), x) = 0$, respectively with $g \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Wang and Tian in [86, 87] studied the existence of solutions for difference equations involving causal operators without impulses with nonlinear boundary conditions to the two following boundary value problems:

$$\begin{aligned} \Delta x(n) &= (Qx)(n), \quad n \in [0, T-1] = \{0, 1, \ldots, T-1\}, \\ g(x(0); x(T)) &= 0, \end{aligned} \tag{3.2}$$

with $\Delta x(n) = x(n+1) - x(n)$, $g \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $E_0 = \mathcal{C}([0, T-1], \mathbb{R})$ and $Q \in \mathcal{C}(E_0, E_0)$ is a causal operator.

$$\begin{aligned} \Delta x(n-1) &= (Qx)(n), \quad n \in [1, T] = \{1, 1, \ldots, T\}, \\ g(x(0); Nx(T)) &= 0, \end{aligned} \tag{3.3}$$

with $\Delta x(n-1) = x(n) - x(n-1)$, $g \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $E_1 = \mathcal{C}([1, T], \mathbb{R})$ and $Q \in \mathcal{C}(E_1, E_1)$ is a causal operator.

In 2006, the authors in [64] obtained the existence of positive solutions for a class of first order impulsive difference equations with periodic boundary value conditions using fixed point theorems of Krasnosel'skii and Leggett Williams, for this problem

$$\begin{aligned} -\Delta x(n) + p(n)x(n) &= f(n, x(n)), \quad n \in [0, T-1],\, n \neq n_k, \\ \Delta x(n_k) &= b_k\, x(n_k), \quad n = n_k, \\ x(0) &= x(T), \end{aligned} \tag{3.4}$$

where $\{p(n)\}_{n=0}^{T-1}, b_k$ are real number sequences and $n_k \in [0, T-1] \cap \mathbb{N}$ are fixed points such that $0 = n_0 < n_1 < \ldots < n_k < n_{k+1} = T - 1\cdot$

3.3 Auxiliary results

Approach used

In this section, we propose a new approach that ensure the existence of at least one positive solution to the BVP (3.1). The nonlinear terms in the equation and in the boundary conditions as well as the jump function satisfy a general polynomial growth conditions. Our existence result is based on a recent fixed point index theory developed by Mebarki et al. in [34, 43] for the sum of two operators on cones of a Banach space. Precisely, our method involves the fixed point index for the sum of two operators $T + F$ on cones of a Banach space, where $I - T$ is Lipschitz invertible and F is a k-set contraction.

Assumptions

We suppose that

$(\mathcal{H}_1)$ The functions f, g, I_k, $k \in \{1, \ldots, p\}$, satisfy

$$\begin{aligned} 0 &\leq f(n, x(n)) \leq a_1(n) + a_2(n)|x(n)|^{p_1}, \\ 0 &\leq g(x(0), x(T)) \leq b_1 + b_2|x(0)|^{p_2} + b_3|x(T)|^{p_3}, \\ 0 &\leq I_k(x(n_k)) \leq a_3(n_k) + a_4(n_k)|x(n_k)|^{p_4}, \quad k \in \{1, \ldots, p\}, \end{aligned}$$

where $a_1, a_2, a_3, a_4 \in \mathcal{C}(J, \mathbb{R})$ are positive functions, b_1, b_2, b_3, p_1, p_2, p_3, p_4 are nonnegative constants, and

$$0 \leq a_1(n), a_2(n), a_3(n), a_4(n), b_1, b_2, b_3 \leq D, \quad n \in J,$$

for some positive constant D.

$(\mathcal{H}_2)$ The constants $c \in (0,1)$, $B > 0, D > 0$, $M > 0$, $N > 0$, $T \in \mathbb{N}$, $p_j \geq 0$, $j \in \{1, \ldots, 4\}$, satisfy

$$M - N(1-c)^T > 0$$

and

$$\begin{aligned} B_1 &= \frac{D\left(1 + B^{p_2} + B^{p_3}\right)}{M - N(1-c)^T} \\ &\quad + 2T\frac{M+N}{M-N(1-c)^T}\left(D(1+B^{p_1}+B^{p_4}) + cB\right) \\ &< B. \end{aligned}$$

Sum formulation

Lemma 3.1. *Suppose that $(\mathcal{H}_1)$ holds. Let $x \in \mathcal{C}(J, \mathbb{R})$ satisfies the equation*

$$\begin{aligned} x(n) &= \frac{g(x(0), x(T))(1-c)^n}{M - N(1-c)^T} + \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\left(f(j, x(j)) + cx(j)\right) \\ &\quad + \sum_{0<n_k\leq T-1} G(n, n_k)\left(cx(n_k) + I_k(x(n_k))\right), \quad n \in J. \end{aligned}$$

Then it satisfies the BVP (3.1).

where $0 < c < 1$ and

$$G(n,j) = \frac{1}{M - N(1-c)^T}\begin{cases} M\frac{(1-c)^n}{(1-c)^{j+1}}, & 0 \leq j \leq n-1, \\ N\frac{(1-c)^{T+n}}{(1-c)^{j+1}}, & n \leq j \leq T-1. \end{cases}$$

Proof. We have

$$\begin{aligned} x(n) &= \frac{g(x(0), x(T))(1-c)^n}{M - N(1-c)^T} + \frac{M(1-c)^n}{M-N(1-c)^T}\sum_{0\leq j\leq n-1, j\neq n_k}\frac{1}{(1-c)^{j+1}}\left(f(j,x(j)) + cx(j)\right) \\ &\quad + \frac{N(1-c)^{T+n}}{M-N(1-c)^T}\sum_{n\leq j\leq T-1, j\neq n_k}\frac{1}{(1-c)^{j+1}}\left(f(j,x(j)) + cx(j)\right) \\ &\quad + \frac{M}{M-N(1-c)^T}\sum_{0<n_k\leq n-1}\frac{(1-c)^n}{(1-c)^{n_k+1}}\left(cx(n_k) + I_k(x(n_k))\right) \\ &\quad + \frac{N}{M-N(1-c)^T}\sum_{n\leq n_k\leq T-1}\frac{(1-c)^{T+n}}{(1-c)^{n_k+1}}\left(cx(n_k) + I_k(x(n_k))\right), \quad n \in J. \end{aligned}$$

Hence, for $n \neq n_k$, $k \in \{1, \dots, p\}$, we have

$$\begin{aligned}
x(n+1) &= \frac{g(x(0), x(T))(1-c)^{n+1}}{M - N(1-c)^T} \\
&\quad + \frac{M(1-c)^{n+1}}{M - N(1-c)^T} \sum_{0 \leq j \leq n, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j, x(j)) + cx(j)\right) \\
&\quad + \frac{N(1-c)^{T+n+1}}{M - N(1-c)^T} \sum_{n+1 \leq j \leq T-1, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j, x(j)) + cx(j)\right) \\
&\quad + \frac{M}{M - N(1-c)^T} \sum_{0 < n_k \leq n} \frac{(1-c)^{n+1}}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right) \\
&\quad + \frac{N}{M - N(1-c)^T} \sum_{n+1 \leq n_k \leq T-1} \frac{(1-c)^{T+n+1}}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right) \\
&= \frac{g(x(0), x(T))(1-c)^{n+1}}{M - N(1-c)^T} + \frac{M}{M - N(1-c)^T} \left(f(n, x(n)) + cx(n)\right) \\
&\quad + \frac{M(1-c)^{n+1}}{M - N(1-c)^T} \sum_{0 \leq j \leq n-1, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j, x(j)) + cx(j)\right) \\
&\quad - \frac{N(1-c)^T}{M - N(1-c)^T} \left(f(n, x(n)) + cx(n)\right) \\
&\quad + \frac{N(1-c)^{T+n+1}}{M - N(1-c)^T} \sum_{n \leq j \leq T-1, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j, x(j)) + cx(j)\right) \\
&\quad + \frac{M(1-c)^{n+1}}{M - N(1-c)^T} \sum_{0 < n_k \leq n-1} \frac{1}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right) \\
&\quad + \frac{N(1-c)^{T+n+1}}{M - N(1-c)^T} \sum_{n \leq n_k \leq T-1} \frac{1}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right) \\
&= (1-c) \Bigg(\frac{g(x(0), x(T))(1-c)^n}{M - N(1-c)^T} \\
&\quad + \frac{M(1-c)^n}{M - N(1-c)^T} \sum_{0 \leq j \leq n-1, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j, x(j)) + cx(j)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{N(1-c)^{T+n}}{M-N(1-c)^T}\sum_{n\le j\le T-1, j\neq n_k}\frac{1}{(1-c)^{j+1}}\left(f(j,x(j))+cx(j)\right)\\
&+\frac{M(1-c)^{n}}{M-N(1-c)^T}\sum_{0<n_k\le n-1}\frac{1}{(1-c)^{n_k+1}}\left(cx(n_k)+I_k(x(n_k))\right)\\
&+\frac{N(1-c)^{T+n}}{M-N(1-c)^T}\sum_{n\le n_k\le T-1}\frac{1}{(1-c)^{n_k+1}}\left(cx(n_k)+I_k(x(n_k))\right)\Bigg)\\
&+f(n,x(n))+cx(n)\\
=\;&(1-c)x(n)+f(n,x(n))+cx(n)\\
=\;&f(n,x(n))+x(n).
\end{aligned}
$$

So,

$$\Delta x(n)=f(n,x(n)),\quad n\neq n_k.$$

Next, for $n=n_k$, $k\in\{1,\ldots,p\}$, we have

$$
\begin{aligned}
\Delta x(n_k) &= x(n_k+1)-x(n_k)\\
&= \frac{g(x(0),x(T))(1-c)^{n_k+1}}{M-N(1-c)^T}\\
&\quad+\frac{M(1-c)^{n_k+1}}{M-N(1-c)^T}\sum_{0\le j\le n_k, j\neq n_l}\frac{1}{(1-c)^{j+1}}\left(f(j,x(j))+cx(j)\right)\\
&\quad+\frac{N(1-c)^{T+n_k+1}}{M-N(1-c)^T}\sum_{n_k+1\le j\le T-1, j\neq n_l}\frac{1}{(1-c)^{j+1}}\left(f(j,x(j))+cx(j)\right)\\
&\quad+\frac{M(1-c)^{n_k+1}}{M-N(1-c)^T}\sum_{0<n_l\le n_k}\frac{1}{(1-c)^{n_l+1}}\left(cx(n_l)+I_k(x(n_l))\right)\\
&\quad+\frac{N(1-c)^{T+n_k+1}}{M-N(1-c)^T}\sum_{n_k+1\le n_l\le T-1}\frac{1}{(1-c)^{n_l+1}}\left(cx(n_l)+I_k(x(n_l))\right)\\
&\quad-x(n_k)
\end{aligned}
$$

$$
\begin{aligned}
&= (1-c)\Bigg(\frac{g(x(0),x(T))(1-c)^{n_k}}{M-N(1-c)^T} \\
&\quad +\frac{M(1-c)^{n_k}}{M-N(1-c)^T}\sum_{0\le j\le n_k-1, j\ne n_l}\frac{1}{(1-c)^{j+1}}\left(f(j,x(j))+cx(j)\right) \\
&\quad +\frac{N(1-c)^{T+n_k}}{M-N(1-c)^T}\sum_{n_k\le j\le T-1, j\ne n_l}\frac{1}{(1-c)^{j+1}}\left(f(j,x(j))+cx(j)\right) \\
&\quad +\frac{M(1-c)^{n_k}}{M-N(1-c)^T}\sum_{0<n_l\le n_k-1}\frac{1}{(1-c)^{n_l+1}}\left(cx(n_l)+I_k(x(n_l))\right) \\
&\quad +\frac{N(1-c)^{T+n_k}}{M-N(1-c)^T}\sum_{n_k\le n_l\le T-1}\frac{1}{(1-c)^{n_l+1}}\left(cx(n_l)+I_k(x(n_l))\right)\Bigg) \\
&\quad +\frac{M}{M-N(1-c)^T}\left(cx(n_k)+I_k(x(n_k))\right) \\
&\quad -\frac{N(1-c)^T}{M-N(1-c)^T}\left(cx(n_k)+I_k(x(n_k))\right)-x(n_k) \\
&= (1-c)x(n_k)+cx(n_k)+I_k(x(n_k))-x(n_k) \\
&= I_k(x(n_k)).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
Mx(0) &= \frac{g(x(0),x(T))M}{M-N(1-c)^T} \\
&\quad +\frac{MN(1-c)^T}{M-N(1-c)^T}\sum_{0\le j\le T-1, j\ne n_k}\frac{1}{(1-c)^{j+1}}\left(f(j,x(j))+cx(j)\right) \\
&\quad +\frac{MN(1-c)^T}{M-N(1-c)^T}\sum_{0\le n_k\le T-1}\frac{1}{(1-c)^{n_k+1}}\left(cx(n_k)+I_k(x(n_k))\right),
\end{aligned}
$$

$$
\begin{aligned}
Nx(T) &= \frac{g(x(0),x(T))N(1-c)^T}{M-N(1-c)^T} \\
&\quad + \frac{MN(1-c)^T}{M-N(1-c)^T} \sum_{0\le j\le T-1, j\neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad + \frac{MN(1-c)^T}{M-N(1-c)^T} \sum_{0\le n_k\le T-1} \frac{1}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right).
\end{aligned}
$$

Therefore

$$Mx(0) - Nx(T) = g(x(0), x(T)).$$

This completes the proof. □

3.4 Main result

Existence result

Our main result is as follows.

Theorem 3.1. *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold. Then the BVP (3.1) has at least one positive solution* $x \in \mathcal{C}(J,\mathbb{R})$ *so that* $0 \le x(n) < B, \quad n \in J.$

To prove this result we will use proposition 1.8 for the case $\theta = 0$.

For $x \in \mathcal{C}(J,\mathbb{R})$, define the operator

$$
\begin{aligned}
Fx(n) &= \frac{g(x(0),x(T))(1-c)^n}{M-N(1-c)^T} + \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\left(f(j,x(j)) + cx(j)\right) \\
&\quad + \sum_{0<n_k\le T-1} G(n,n_k)\left(cx(n_k) + I_k(x(n_k))\right), \ n \in J.
\end{aligned}
$$

By Lemma 3.1, it follows that any fixed point $x \in \mathcal{C}(J,\mathbb{R})$ of the operator F is a solution to the BVP (3.1).

Auxiliary lemmas

We have that

$$G(n,j) \le \frac{M+N}{M-N(1-c)^T}, \quad n, j \in J. \tag{3.5}$$

In the Banach space $\mathcal{C}(J,\mathbb{R})$ of the continuous real-valued functions defined on J, define the norm

$$\|x\| = \max_{n\in J} |x(n)|.$$

Lemma 3.2. *Suppose that $(\mathcal{H}_1)$ holds. If $x \in \mathcal{C}(J,\mathbb{R})$, $\|x\| \le B$, then*

$$\begin{aligned} 0 &\le f(n,x(n)) \le D\left(1+B^{p_1}\right), \, n \in J, \\ 0 &\le g(x(0),x(T)) \le D\left(1+B^{p_2}+B^{p_3}\right), \\ 0 &\le I_k(x(n_k)) \le D\left(1+B^{p_4}\right), \quad k \in \{1,\dots,p\}. \end{aligned}$$

Proof. By $(\mathcal{H}1)$, we get

$$\begin{aligned} 0 &\le f(n,x(n)) \\ &\le a_1(n) + a_2(n)|x(n)|^{p_1} \\ &\le D\left(1+B^{p_1}\right), \, n \in J, \end{aligned}$$

and

$$\begin{aligned} 0 &\le g(x(0),x(T)) \\ &\le b_1 + b_2|x(0)|^{p_2} + b_3|x(T)|^{p_3} \\ &\le D\left(1+B^{p_2}+B^{p_3}\right), \end{aligned}$$

and

$$\begin{aligned} 0 &\le I_k(x(n_k)) \\ &\le a_3(n_k) + a_4(n_k)|x(n_k)|^{p_4} \\ &\le D\left(1+B^{p_4}\right), \quad k \in \{1,\dots,p\}. \end{aligned}$$

This completes the proof. □

Lemma 3.3. *Suppose that $f \in \mathcal{C}(J\times\mathbb{R})$, $g \in \mathcal{C}(\mathbb{R}\times\mathbb{R})$ and $I_k \in \mathcal{C}(\mathbb{R})$, $k \in \{1,\dots,p\}$. Then $F : \mathcal{C}(J,\mathbb{R}) \to \mathcal{C}(J,\mathbb{R})$ is a continuous operator.*

Proof. (a) Since $G \in \mathcal{C}(J \times J)$, $f \in \mathcal{C}(J \times \mathbb{R})$, $g \in \mathcal{C}(\mathbb{R} \times \mathbb{R})$ and $I_k \in \mathcal{C}(\mathbb{R})$, $k \in \{1, \dots, p\}$, the operator F maps $\mathcal{C}(J, \mathbb{R})$ into $\mathcal{C}(J, \mathbb{R})$.

(b) F is continuous. In fact, take $\{x_l\}_{l \in \mathbb{N}} \subset \mathcal{C}(J, \mathbb{R})$ such that $x_l \to x$, as $l \to +\infty$ in $\mathcal{C}(J, \mathbb{R})$. Fix $\varepsilon > 0$ arbitrarily. Then there is a $\delta = \delta(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} |x_l(n) - x(n)| &< \varepsilon, \\ |f(n, x_l(n)) - f(n, x(n))| &< \varepsilon, \\ |I_k(x_l(n)) - I_k(x(n))| &< \varepsilon \end{aligned}$$

for any $n \in J$, $k \in \{1, \dots, p\}$, and for any $l \geq \delta$. We have

$$\begin{aligned} |Fx_l(n) - Fx(n)| &= \Bigg| \frac{(g(x_l(0), x_l(T)) - g(x(0), x(T)))\,(1-c)^n}{M - N(1-c)^T} \\ &\quad + \sum_{j=0, j \neq n_k}^{T-1} G(n, j)\left((f(j, x_l(j)) - f(j, x(j))) + c(x_l(j) - x(j))\right) \\ &\quad + \sum_{0 < n_k \leq T-1} G(n, n_k)\left(c(x_l(n_k) - x(n_k)) + (I_k(x_l(n_k)) - I_k(x(n_k)))\right) \Bigg| \\ &\leq \frac{|g(x_l(0), x_l(T)) - g(x(0), x(T))|}{M - N(1-c)^T} \\ &\quad + \sum_{j=0, j \neq n_k}^{T-1} G(n, j)\left(|f(j, x_l(j)) - f(j, x(j))| + c|x_l(j) - x(j)|\right) \\ &\quad + \sum_{0 < n_k \leq T-1} G(n, n_k)\left(c|x_l(n_k) - x(n_k)| + |I_k(x_l(n_k)) - I_k(x(n_k))|\right) \\ &< \frac{\varepsilon}{M - N(1-c)^T} \\ &\quad + \sum_{j=0, j \neq n_k}^{T-1} \frac{M+N}{M - N(1-c)^T}\,(\varepsilon + c\,\varepsilon) \\ &\quad + \sum_{0 < n_k \leq T-1} \frac{M+N}{M - N(1-c)^T}\,(\varepsilon + c\,\varepsilon) \\ &\leq \varepsilon \Bigg(\frac{1}{M - N(1-c)^T} \\ &\quad + 2T \frac{M+N}{M - N(1-c)^T}\,(1+c) \Bigg), \quad n \in J, \quad l \geq \delta. \end{aligned}$$

This completes the proof. ∎

Lemma 3.4. *Suppose that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. For $x \in \mathcal{C}(J, \mathbb{R})$, $\|x\| \leq B$, we have*

$$Fx(n) \leq B_1, \quad |\Delta Fx(n)| \leq 2B_1, \quad n \in J.$$

Proof. We have

$$\begin{aligned}
Fx(n) &\leq \frac{D\left(1+B^{p_2}+B^{p_3}\right)}{M-N(1-c)^T} \\
&\quad + \sum_{j=0, j\neq n_k}^{T-1} \frac{M+N}{M-N(1-c)^T}\left(D\left(1+B^{p_1}+B^{p_4}\right)+cB\right) \\
&\quad + \sum_{0<n_k\leq T-1} \frac{M+N}{M-N(1-c)^T}\left(D\left(1+B^{p_1}+B^{p_4}\right)+cB\right) \\
&\leq \frac{D\left(1+B^{p_2}+B^{p_3}\right)}{M-N(1-c)^T} \\
&\quad +2T\frac{M+N}{M-N(1-c)^T}\left(D(1+B^{p_1}+B^{p_4})+cB\right) \\
&= B_1, \quad n\in J.
\end{aligned}$$

Next,

$$\begin{aligned}
|\Delta Fx(n)| &= |Fx(n+1)-Fx(n)| \\
&\leq Fx(n+1)+Fx(n) \\
&\leq 2B_1, \quad n\in J.
\end{aligned}$$

This completes the proof. □

Proof of the main result

Take $\epsilon>0$ arbitrarily. Let $E=\mathcal{C}(J,\mathbb{R})$ be endowed with the norm $\|x\|=\max\limits_{n\in J}|x(n)|$, and

$$\begin{aligned}
\mathcal{P} &= \{x\in E : x(n)\geq 0, \quad n\in J\}, \\
\Omega &= \mathcal{P}_{2B}=\{x\in\mathcal{P} : \|x\|<2B\}, \\
U &= \mathcal{P}_B=\{x\in\mathcal{P} : \|x\|<B\}.
\end{aligned}$$

For $x\in E$, define the operators

$$\begin{aligned}
T_1x(n) &= (1+\epsilon)x(n), \\
F_1x(n) &= -\epsilon Fx(n), \quad n\in J.
\end{aligned}$$

Note that for any fixed point $x\in E$ of the operator T_1+F_1 is a solution of the BVP (3.1).

1. For $x, y \in E$, we have

$$\|(I - T_1)^{-1}x - (I - T_1)^{-1}y\| = \frac{1}{\epsilon}\|x - y\|,$$

   i.e., $(I - T_1) : E \to E$ is Lipschitz invertible with constant $\frac{1}{\epsilon}$.

2. According to the Ascoli-Arzelà compactness criteria, by Lemma 3.3 and Lemma 3.4, it follows that $F_1 : \overline{U} \to E$ is a completely continuous operator. Therefore $F_1 : \overline{U} \to E$ is a 0-set contraction.

3. Let $t \in [0, 1]$ and $x \in \overline{U}$ be arbitrarily chosen. Then

$$z = tFx \in E$$

   and

$$\begin{aligned} z(n) &\leq tB_1 \\ &< tB \\ &\leq B, \quad n \in J, \end{aligned}$$

   i.e., $z \in \Omega$. Next,

$$\begin{aligned} tF_1x(n) &= -t\epsilon Fx(n) \\ &= -\epsilon z(n) \\ &= (I - T_1)z(n), \quad n \in J. \end{aligned}$$

   Thus, $tF_1(\overline{U}) \subset (I - T_1)(\Omega)$.

4. Note that

$$(I - T_1)^{-1}0 = 0 \in U.$$

5. Assume that there are $x \in \partial U \bigcap \Omega$ and $\lambda \in [0, 1]$ such that

$$(I - T_1)x = \lambda F_1 x.$$

   If $\lambda = 0$, then

$$0 = (I - T_1)x = -\epsilon x \quad \text{on} \quad J,$$

whereupon $x(n) = 0$, $n \in J$, thus a contradiction because $x \in \partial U$. Therefore $\lambda \in (0, 1]$. Let $n_1 \in J$ be such that $x(n_1) = B$. Then

$$\begin{aligned}(I - T_1)x(n_1) &= -\epsilon x(n_1)\\ &= -\epsilon B\\ &= -\epsilon \lambda F x(n_1),\end{aligned}$$

whereupon

$$\begin{aligned}B &= \lambda F x(n_1)\\ &\leq \lambda B_1\\ &< \lambda B\\ &\leq B,\end{aligned}$$

i.e., $B < B$, which is a contradiction.

Consequently, by setting $\theta = 0$ in Proposition 1.8 and the existence property of the fixed point index, it follows that the operator $T_1 + F_1$ has a fixed point in U, denote it by x. We have

$$0 \leq x(n) < B, \quad n \in J,$$

and $x \in E$ is a solution of the BVP (3.1).

3.5 Example

Consider the following boundary value problem:

$$\begin{aligned}\Delta x(n) &= \frac{(x(n))^2}{10^{10000}(n^2+1)}, \quad n \in [0, 20],\\ \Delta x(n_k) &= \frac{(x(n_k))^2}{10^{10000}}, \quad k \in \{1, 2, 3, 4\},\\ 10^{100}x(0) - x(20) &= \frac{(x(0))^2}{10^{10000}(1 + x(20) + (x(20))^2)}\end{aligned}$$

Let

$$D = \frac{1}{10^{10000}}, \quad B = 1, \quad p = 4, \quad p_1 = p_2 = p_3 = p_4 = 2, \quad T = 20,$$

and

$$a_1(n) = a_2(n) = a_3(n) = a_4(n) = \frac{1}{10^{10000}}, \quad n \in [0, 20], \quad b_1 = b_2 = b_3 = \frac{1}{10^{10000}},$$

$$n_1 = 1, \quad n_2 = 3, \quad n_3 = 7, \quad n_4 = 11,$$

and

$$N = 1, \quad c = \frac{1}{10^{10000}}, \quad M = 10^{100}.$$

Then

$$B_1 = \frac{\frac{3}{10^{10000}}}{10^{100} - \left(1 - \frac{1}{10^{10000}}\right)^{20}} + 40\frac{10^{100}+1}{10^{100} - \left(1 - \frac{1}{10^{10000}}\right)^{20}}\left(\frac{4}{10^{10000}}\right) < 1 = B$$

Hence, by Theorem 3.1, we obtain the existence of a solution $x \in \mathcal{C}([0, 20] \cap \mathbb{N}, \mathbb{R})$ such that

$$0 \leq x(n) < 1, \quad n \in \{0, 1, \ldots, 20\}.$$

3.6 Comparison and conclusion

(1) The boundary conditions considered in this work involving nonlinear functional at two point are more general. They include, as particular cases, periodic, multipoint boundary value conditions and integral boundary value conditions. The case of an initial value problem is not considered because $N > 0, M > 0$.

(2) In [85], the BVP (3.1) is investigated when

$$f(n, x) - f(n, y) \geq -L(x - y)$$

for $\alpha_0 \leq \alpha(n) \leq y \leq x \leq \beta(n) \leq \beta_0$, $n \in J$, and

$$I_k(x) - I_k(y) \geq -L_k(x - y)$$

for $\alpha_0 \leq \alpha(n_k) \leq y \leq x \leq \beta(n_k) \leq \beta_0$, $k \in \{1, \ldots, p\}$, and g is a constant, where α_0, β_0 are nonnegative constants, α and β are suitable nonnegative functions, $0 < L, L_k < 1$, $k \in \{1, \ldots, p\}$. It is given in [85] a criteria for existence of positive minimal and maximal solutions. If

$$f(n, x) = I_k(x) = \frac{a}{(1 + x)^2}, \quad x \geq 0, \quad a > \frac{(1 + \beta_0)^4}{1 + \alpha_0},$$

Then

$$
\begin{aligned}
I_k(x) - I_k(y) &= f(n,x) - f(n,y) \\
&= \frac{a}{(1+x)^2} - \frac{a}{(1+y)^2} \\
&= -\frac{a(x-y)(2+x+y)}{(1+x)^2(1+y)^2} \\
&\le -\frac{2a(x-y)(1+\alpha_0)}{(1+\beta_0)^4} \\
&< -2(x-y), \quad \alpha_0 \le y \le x \le \beta_0.
\end{aligned}
$$

Thus, the conditions in [85] are not fulfilled, but our conditions hold. Also, our main result is valid in the case when g is not a constant. Therefore we can consider our main result as a complementary and improvement result to those in [85].

4 New fixed point theorem and application to nonlinear second order difference equations

The results of this chapter are obtained by Bouchal, Mebarki and Goergiev in [25].

4.1 Introduction

In this chapter, by making use of the generalized fixed point index developed in [34] we have obtained a new functional fixed point theorem for the operator sum $T+S$, where $I-T$ is Lipschitz invertible and S is a k-set contraction. Then we present a technique that takes advantage of the flexibility of this new fixed point theorem to establish the existence of at least one positive solution for a conjugate boundary value problem for the second order difference equation. Throughout this chapter, $\mathcal{P}$ will refer to a cone in a Banach $(E, \|\,\|)$.

4.2 Functional type fixed point theorems

4.2.1 Historical notes and motivations

A variant of fixed point theorems that have many applications in proving the existence and multiplicity of positive solutions of boundary value problems are fixed point theorems of functional types. This class of theorems originates with Krasnosel'skii fixed point theorem in 1964 [59] where the functional used was the norm and the fixed point is localized in a conical shell of the form $\{x \in \mathcal{P},\, a \leq \|x\| \text{ and } \|y\| \leq b\}$ for $0 < a < b$, but the pionniers of functional fixed point theorem can be traced back to both Leggett and Williams in 1979 [63] where they replaced the norm used in the lower boundary of Guo-Krasnosel'skii fixed point theorem [46, 48] by a positive concave functional of the form $a \leq \alpha(x)$ and then fixed points are localized in sets of the form $\mathcal{P}(\alpha, a, b) = \{x \in \mathcal{P},\ a \leq \alpha(x) \text{ and } \|x\| \leq b\}$.

Later, a slight modification of this theorem is given (see [9, Theorem 16]) by introducing a convex functional β instead of the norm in the upper boundary to have more flexibility in the upper wedge condition, and then fixed points are localized in sets of the form

$$\mathcal{P}(\beta, \alpha, a, b) = \{x \in \mathcal{P},\ a \leq \alpha(x) \text{ and } \beta(x) \leq b\}.$$

In 2002, Avery and Anderson generalized the Guo-Krasnosel'skii fixed point theorem (see [5, Theorem 10]). A generalization that allows the user to choose two functionals that satisfy certain conditions that are used instead of the norm. The interesting point in their result is that these functionals do not need to be concave or convex, which leaves more freedom and flexibility, especially in applications to boundary value problems. This is one of the reasons that motivated us to extend Avery-Anderson's theorem for the sum of two operators.

4.2.2 Functional fixed point theorem for sums of two operators

In the sequel, we will establish an extension of [5, Theorem 10] which guarantees the existence of at least one nontrivial positive solution to some equations of the form $Tx + Sx = x$ posed on cones of a Banach space. The proof is based on the properties of the generalized fixed point index i_* presented in chapter 1.

A map α is said to be a nonnegative continuous functional on a cone $\mathcal{P}$ of a real Banach space E if $\alpha : \mathcal{P} \to [0, \infty)$ is continuous. Let α and β be nonnegative continuous functionals on $\mathcal{P}$; and let r, R be two positive real numbers, we define the sets:

$$\mathcal{P}(\beta, R) = \{x \in \mathcal{P} :\ \beta(x) < R\},$$

$$\mathcal{P}(\beta, \alpha, r, R) = \{x \in \mathcal{P} :\ r < \alpha(x) \text{ and } \beta(x) < R\}. \tag{4.1}$$

Theorem 4.1. *Let E be a Banach space; $\mathcal{P} \subset E$ a cone; α and β be nonnegative continuous functionals on $\mathcal{P}$ and let $r < R$ be two positive real numbers. Let $T : \Omega \subset \mathcal{P} \to E$ be a mapping* *such that* $(I-T)$ *is Lipschitz invertible with constant* $\gamma > 0$ *and* $S : \overline{\mathcal{P}(\beta, R)} \to E$ *be a* k*-set contraction mapping with* $0 \leq k < \gamma^{-1}$*. Assume that* $\mathcal{P}(\beta,\alpha,r,R) \cap \Omega \neq \emptyset$, $\overline{\mathcal{P}(\alpha,r)} \subset \mathcal{P}(\beta,R)$ *and*

$$S(\overline{\mathcal{P}(\beta,R)}) \subset (I-T)(\Omega). \tag{4.2}$$

If one of the two following conditions is satisfied

$(\mathcal{A}_1)$ *for all* $x \in \partial\mathcal{P}(\alpha,r)$ *and* $\lambda > 1$ *with* $\lambda x \in \Omega$ *and* $T(\lambda x) + Sx \in \mathcal{P}$,

$$\alpha(T(\lambda x) + Sx) \leq r, \ \ \lambda\alpha(x) \leq \alpha(\lambda x) \quad \text{and} \ \ \alpha(0) < r, \tag{4.3}$$

and there exists $u_0 \in \mathcal{P}^*$, *for all* $\eta > 0$ *and* $x \in \partial\mathcal{P}(\beta,R) \cap (\Omega + \eta u_0)$ *with* $T(x - \eta u_0) + Sx + \eta u_0 \in \mathcal{P}$,

$$\beta(T(x-\eta u_0) + Sx + \eta u_0) \neq R, \tag{4.4}$$

or

$(\mathcal{A}_2)$ *for all* $x \in \partial\mathcal{P}(\beta,R)$ *and* $\lambda > 1$ *with* $\lambda x \in \Omega$ *and* $T(\lambda x) + Sx \in \mathcal{P}$,

$$\beta(T(\lambda x) + Sx) \leq R, \ \ \lambda\beta(x) \leq \beta(\lambda x) \quad \text{and} \ \ \beta(0) < R, \tag{4.5}$$

and there exists $u_0 \in \mathcal{P}^*$, *for all* $\eta > 0$ *and* $x \in \partial\mathcal{P}(\alpha,r) \cap (\Omega + \eta u_0)$ *with* $T(x - \eta u_0) + Sx + \eta u_0 \in \mathcal{P}$,

$$\alpha(T(x-\eta u_0) + Sx + \eta u_0) \neq r, \tag{4.6}$$

then $T+S$ *has at least one nontrivial fixed point* $x^* \in \overline{\mathcal{P}(\beta,\alpha,r,R)} \cap \Omega$.

Proof. Suppose that $Tx + Sx \neq x$ for all $x \in \partial\mathcal{P}(\beta,\alpha,r,R)$, otherwise we are finished.

We will suppose that the condition $(\mathcal{A}_1)$ holds; the proof when $(\mathcal{A}_2)$ is satisfied is similar.

Claim 1: $Sx \neq (I-T)(\lambda x)$ for all $x \in \partial\mathcal{P}(\alpha,r)$, $\lambda > 1$ and $\lambda x \in \Omega$.

On the contrary, suppose that there exists a $x_0 \in \partial\mathcal{P}(\alpha,r)$, $\lambda_0 > 1$ and $\lambda_0 x_0 \in \Omega$ such that $T(\lambda_0 x_0) + Sx_0 = \lambda_0 x_0$. Then,

$$r \geq \alpha\left(T(\lambda_0 x_0) + Sx_0\right) = \alpha\left(\lambda_0 x_0\right) \geq \lambda_0\alpha(x_0) > \alpha(x_0) = r,$$

which is a contradiction with (4.3).

Note that $0 \in \mathcal{P}(\alpha, r)$. Hence, from Proposition 1.6,

$$i_*(T+S, \mathcal{P}(\alpha, r) \cap \Omega, \mathcal{P}) = 1.$$

Claim 2: $Sx \neq (I-T)(x-\eta u_0)$ for all $\eta > 0$ and $x \in \partial\mathcal{P}(\beta, R) \cap (\Omega + \eta u_0)$, for some $u_0 \in \mathcal{P}^*$.

On the contrary, for any $u_0 \in \mathcal{P}^*$ there exist $\eta_0 > 0$ and $z_0 \in \partial\mathcal{P}(\beta, R) \cap (\Omega + \eta u_0)$ such that $Sz_0 = (I-T)(z_0 - \eta_0 u_0)$. So,

$$T(z_0 - \eta_0 u_0) + Sz_0 + \eta_0 u_0 = z_0.$$

Then,

$$\beta(T(z_0 - \eta_0 u_0) + Sz_0 + \eta_0 u_0) = \beta(z_0) = R,$$

which is a contradiction with (4.4).

As a result of Proposition 1.9, we arrive at

$$i_*(T+S, \mathcal{P}(\beta, R) \cap \Omega, \mathcal{P}) = 0.$$

Thus, from the additivity property of the fixed point index, we have

$$\begin{aligned} i_*(T+S, \mathcal{P}(\beta, \alpha, r, R) \cap \Omega, \mathcal{P}) &= i_*(T+S, \mathcal{P}(\beta, R) \cap \Omega, \mathcal{P}) - i_*(T+S, \mathcal{P}(\alpha, r) \cap \Omega, \mathcal{P}) \\ &= -1. \end{aligned}$$

By the existence property of the fixed point index the operator $T+S$ has at least one fixed point $x^* \in \mathcal{P}(\beta, \alpha, r, R) \cap \Omega$. Hence the desired result. □

4.3 Application to difference equations

In this section, we will investigate the difference equation

$$\triangle^2 u(k) + f(k, u(k)) = 0, \quad k \in \{0, 1, \dots, N\}, \; N \in \mathbb{N}, N > 1, \tag{4.7}$$

with boundary conditions

$$u(0) = u(N+2) = 0, \tag{4.8}$$

where $f : \{0, ..., N+2\} \times [0, \infty) \to [0, \infty)$ is a continuous function.

Δ^2 is the second forward difference operator which acts on u by $\triangle^2 u(k) = u(k+2) - 2u(k+1) + u(k)$, $k \in \{0, 1, \ldots, N\}$, with $\{0, 1, ..., N\}$ is a discrete interval. By positive solution, we mean a function $u : \{0, \ldots, N+2\} \to \mathbb{R}$ such that $u(k) \geq 0$ on $\{0, 1, \ldots, N+2\}$ and verifies the posed BVP.

In [5] (2002), Anderson and Avery prove the existence of a positive solution to the autonomous problem given below, using the cone compression and expansion fixed point theorem of functional type [5, Theorem 10].

$$\triangle^2 u(k-1) + f(u(k)) = 0, \quad k \in \{a+1, \ldots, b+1\},\ a, b \in \mathbb{N}, b > a+2, \tag{4.9}$$

with boundary conditions

$$u(a) = u(b+2) = 0, \tag{4.10}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative function.

In [74] (2012), Neugebaeur and Seelbach give an application to an extension of the Leggett-Williams fixed point theorem due to Avery [13, Theorem 3.1], to obtain at least one positive solution to the difference equation

$$\triangle^2 u(k) + f(u(k)) = 0, \quad k \in \{0, \ldots, N-2\}, \tag{4.11}$$

with boundary conditions

$$u(0) = u(N) = 0, \tag{4.12}$$

where $f : [0, \infty) \to [0, \infty)$ is a continuous function.

In [75] (2017), a compression-expansion fixed point theorem of functional type has been used to obtain at least one positive solution for the autonomous second order difference equation:

$$\triangle^2 u(k) + f(u(k)) = 0, \quad k \in \{0, 1, \ldots, N\},\ N \in \mathbb{N}, N > 1, \tag{4.13}$$

with boundary conditions

$$u(0) = u(N+2) = 0, \tag{4.14}$$

where $f : [0, \infty) \to [0, \infty)$ is a continuous function satisfying some conditions of monotonic type. In [76] (2020), the layered compression-expansion fixed point theorem was applied to show the existence of solutions to the BVP (4.13)-(4.14), where the nonlinearity f is the sum of a monotonic increasing and a monotonic decreasing functions.

In these three last works the functions in the cone are required to be nonnegative, nondecreasing on the half of the interval and symetric when applying the Avery, Anderson and Henderson theorems and they obtain the localisation of the positive solution and its symmetry.

In what follows, by using our approach, we will establish sufficient criteria for the existence of positive solutions to BVP (4.7)-(4.8) on a cone which is nonnegative.

4.3.1 Auxiliary results

Define the function

$$H(k,l) = \frac{1}{N+2}\begin{cases} k(N+2-l), & k \in \{0,\dots,l\}, \\ l(N+2-k), & k \in \{l+1,\dots,N+2\}, \end{cases}$$

for any $l \in \{0,...,N+2\}$ is the Green's function for $-\Delta^2 u = 0$ satisfying the boundary conditions (4.8). In [75] it is proved that if $u \in E$ is a solution to the BVP (4.7)-(4.8), then it is a solution to the sum equation

$$u(k) = \sum_{l=1}^{N+1} H(k,l) f(l,u(l)), \quad k \in \{0,\dots,N+2\},$$

and conversely. We have that

$$H(k,l) \le N+2, \quad k,l \in \{0,\dots,N+2\}.$$

Assumptions

Suppose that we have the following hypothesis

$$(\mathcal{H}_1) \begin{cases} 0 \le f(k,u(k)) \le a(k) + b(k)|u(k)|^p,\ p \ge 0,\ a,b : \{0,\dots,N+2\} \to [0,\infty) \text{ be such that} \\ 0 \le a(k),\ b(k) \le B,\ k \in \{0,\dots,N+2\}, \text{ for some positive constant } B. \end{cases}$$

$(\mathcal{H}_2)$ ϵ, A_1, B, B_1, R, R_1, r are positive constants such that

$$\epsilon \in (0,1), \quad \frac{B_1}{2} > A_1(N+3)\left((N+2)(N+1)B\left(1+R^p\right)+R\right),$$

$$\frac{r}{A_1} < R, \quad R_1 > \max\{R,1\}, \quad A_1 \in (0,1),$$

$$A_1(\epsilon + r + 2B_1) \le r.$$

Define the Banach space

$$E = \{u : \{0, \dots, N+2\} \to \mathbb{R}\}$$

with the norm

$$\|u\| = \max_{k\in\{0,\dots,N+2\}} |u(k)|.$$

Let

$$S_1 u(k) = \sum_{l=1}^{N+1} H(k,l) f(l, u(l)), \quad k \in \{0, \dots, N+2\}.$$

Auxiliary lemmas

Lemma 4.1. *Suppose that* $(\mathcal{H}_1)$ *holds. Let* $u \in E$ *and* $\|u\| \leq Q$ *for some positive constant* Q*. Then*

$$S_1 u(k) \leq (N+2)(N+1)B(1+Q^p), \quad k \in \{0, \dots, N+2\}.$$

Proof. We have

$$\begin{aligned} S_1 u(k) &= \sum_{l=1}^{N+1} H(k,l) f(l, u(l)) \\ &\leq (N+2) \sum_{l=1}^{N+1} (a(l) + b(l)|u(l)|^p) \\ &\leq (N+2)(N+1)B(1+Q^p), \quad k \in \{0, \dots, N+2\}. \end{aligned}$$

This completes the proof. □

For $u \in E$, define the operator

$$S_2 u(k) = A_1 \sum_{m=0}^{k-1} \left(S_1 u(m) - u(m)\right), \quad k \in \{1, \dots, N+2\}.$$

Lemma 4.2. *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold. Let* $u \in E$ *and*

$$S_2 u(k) = C, \quad k \in \{1, \dots, N+2\}, \tag{4.15}$$

where C *is a constant. Then* u *is a solution to the BVP* (4.7)-(4.8).

Proof. We have

$$\sum_{m=0}^{k-1} \left(\sum_{l=1}^{N+1} H(m,l) f(l, u(l)) - u(m) \right) - \frac{C}{A_1} = 0, \quad k \in \{1, \dots, N+2\}.$$

We take the Δ-operator of both sides of the last equation and we find

$$\begin{aligned} & \sum_{m=0}^{k}\left(\sum_{l=1}^{N+1} H(m,l)f(l,u(l)) - u(m)\right) - \sum_{m=0}^{k-1}\left(\sum_{l=1}^{N+1} H(m,l)f(l,u(l)) - u(m)\right) \\ = & \sum_{l=1}^{N+1} H(k,l)f(l,u(l)) - u(k) = 0, \end{aligned}$$

$k \in \{1, \dots, N+2\}$. This completes the proof. $\square$

Lemma 4.3. *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold. Let* $u \in E$ *and* $\|u\| \le Q$ *for some positive constant* Q*. Then* $\|S_2 u\| \le A_1(N+3)\left((N+2)(N+1)B(1+Q^p)+Q\right)$.

Proof. We have

$$\begin{aligned} \|S_2 u\| & \le A_1 \sum_{m=0}^{k-1} (\|S_1 u\| + \|u\|) \\ & \le A_1 \sum_{m=0}^{N+2} (\|S_1 u\| + \|u\|) \\ & \le A_1(N+3)\left((N+2)(N+1)B(1+Q^p)+Q\right). \end{aligned}$$

This completes the proof. $\square$

4.3.2 Main result

Our main existence result is the following

Theorem 4.2. *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold. Then the BVP* (4.7)-(4.8) *has at least one positive solution* $u^* \in E$ *such that* $\frac{r}{A_1} \le \max_{k\in\{0,\dots,N+2\}} u^*(k) \le R$.

Proof. Let

$$\begin{aligned} \mathcal{P} & = \{u \in E : u \ge 0\}, \\ \Omega & = \mathcal{P}. \end{aligned}$$

For $u \in \mathcal{P}$, define the functionals

$$\begin{aligned} \alpha(u) & = A_1 \max_{k\in\{0,\dots,N+2\}} u(k), \\ \beta(u) & = \max_{k\in\{0,\dots,N+2\}} u(k), \end{aligned}$$

and for $u \in E$, define the operators

$$\begin{aligned} Tu(k) &= -\epsilon\frac{u(k)}{R_1+u(k)}, \\ S_3u(k) &= \epsilon\frac{u(k)}{R_1+u(k)}+u(k)+S_2u(k), \\ Su(k) &= S_3u(k)+B_1, \quad k\in\{1,\ldots,N+2\}. \end{aligned}$$

Note that if $u \in \mathcal{P}$ is a fixed point of the operator $T+S$, then $Tu+Su=u$, whereupon $S_2u(k)=-B_1$, $k\in\{0,\ldots,N+2\}$, and then it is a positive solution to the BVP (4.7)-(4.8).

1. Define the function

$$g(x)=\frac{x}{R_1+x}, \quad x\geq 0.$$

   Then

$$g'(x)=\frac{R_1}{(R_1+x)^2}, \quad x\geq 0,$$

   and

$$|g'(x)|\leq 1, \quad x\geq 0.$$

   Hence,

$$|g(x)-g(y)|\leq |x-y|, \quad x,y\geq 0,$$

   and

$$\left\|\frac{u}{R_1+u}-\frac{v}{R_1+v}\right\|\leq \|u-v\|, \quad u,v\in\mathcal{P}.$$

   Therefore, for $u,v\in\mathcal{P}$, we have

$$\begin{aligned} \|(I-T)u-(I-T)v\| &\geq \|u-v\|-\epsilon\left\|\frac{u}{R_1+u}-\frac{v}{R_1+v}\right\| \\ &\geq (1-\epsilon)\|u-v\|. \end{aligned}$$

   Thus, $I-T:\mathcal{P}\to E$ is Lipschitz invertible with a constant $\gamma=(1-\epsilon)^{-1}$.

2. Let $u\in\overline{\mathcal{P}(\beta,R)}$. Then $\|u\|\leq R$ and by Lemma 4.3, it follows

$$\|S_2u\| \leq A_1(N+3)\left((N+2)(N+1)B(1+R^p)+R\right),$$

   and

$$\epsilon\frac{u(k)}{R_1+u(k)}\leq\epsilon, \quad k\in\{0,\ldots,N+2\}.$$

Consequently

$$\|Su\| \leq \epsilon + R + A_1(N+3)\left((N+2)(N+1)B(1+R^p)+R\right)+B_1,$$

Therefore, $S : \overline{\mathcal{P}(\beta,R)} \to E$ is a completely continuous operator. Thus, S is a 0-set contraction.

3. Because $\frac{r}{A_1} < R$, we have that $\mathcal{P}(\beta,\alpha,r,R)\bigcap\Omega \neq \emptyset$ and $\overline{\mathcal{P}(\alpha,r)} \subset \mathcal{P}(\beta,R)$.

4. Let $u \in \overline{\mathcal{P}(\beta,R)}$ be arbitrarily chosen. By Lemma 4.3 and $(\mathcal{H}_2)$, we find

$$\begin{aligned}\|S_2u\| &\leq A_1(N+3)\left((N+2)(N+1)B(1+R^p)+R\right)\\ &< \frac{B_1}{2}.\end{aligned}$$

Therefore

$$S_2u(k)+\frac{B_1}{2} > 0, \quad k \in \{1,\ldots,N+2\}.$$

Now, using that $u(k) \geq 0$, $k \in \{1,\ldots,N+2\}$, we obtain

$$\begin{aligned}S_3u(k)+\frac{B_1}{2} &= \epsilon\frac{u(k)}{R_1+u(k)}+u(k)+S_2u(k)+\frac{B_1}{2}\\ &\geq S_2u(k)+\frac{B_1}{2}\\ &> 0, \quad k \in \{1,\ldots,N+2\}.\end{aligned}$$

Hence,

$$\begin{aligned}Su(k) &= S_3u(k)+\frac{B_1}{2}+\frac{B_1}{2}\\ &> \frac{B_1}{2}, \quad k \in \{1,\ldots,N+2\}.\end{aligned}$$

Next,

$$\begin{aligned}Su(k) &= \epsilon\frac{u(k)}{R_1+u(k)}+S_2u(k)+u(k)+B_1\\ &\leq \epsilon+R+B_1+B_1\\ &= \epsilon+R+2B_1, \quad k \in \{1,\ldots,N+2\}.\end{aligned}$$

Take

$$v = \frac{-(R_1+\epsilon-Su)+\sqrt{(R_1+\epsilon-Su)^2+4R_1Su}}{2}.$$

We have $v \geq 0$ and therefore $v \in \Omega$. Also,

$$0 = v^2 + (R_1 + \epsilon - Su)v - R_1 Su,$$

whereupon

$$v^2 + R_1 v + \epsilon v = Su\, v + R_1 Su$$

and

$$v(R_1 + v) + \epsilon v = Su(v + R_1).$$

Thus,

$$\begin{aligned} Su &= v + \frac{\epsilon v}{R_1 + v} \\ &= (I - T)v. \end{aligned}$$

Therefore

$$S\left(\overline{\mathcal{P}(\beta, R)}\right) \subset (I - T)(\Omega).$$

5. Let $x \in \partial\mathcal{P}(\alpha, r)$ and $\lambda > 1$. Then

$$\begin{aligned} \alpha(T(\lambda x) + Sx) &= A_1 \max_{k \in \{0,\ldots,N+2\}} \left(-\frac{\epsilon \lambda x(k)}{R_1 + \lambda x(k)} + Sx(k) \right) \\ &\leq A_1 \max_{k \in \{0,\ldots,N+2\}} Sx(k) \\ &\leq A_1(\epsilon + r + 2B_1) \\ &\leq r. \end{aligned}$$

6. For any $x \in \partial\mathcal{P}(\alpha, r)$, $\lambda > 1$, we have

$$\begin{aligned} \alpha(\lambda x) &= A_1 \max_{k \in \{0,\ldots,N+2\}} (\lambda x(k)) \\ &= A_1 \lambda \max_{k \in \{0,\ldots,N+2\}} x(k) \\ &= \lambda\, \alpha(x) \end{aligned}$$

and

$$\alpha(0) < r.$$

7. Let $\eta > 0$ and $u_0 \in \mathcal{P}^\star$ be arbitrarily chosen. Take

$$x \in \partial\mathcal{P}(\beta, R) \bigcap (\Omega + \eta u_0).$$

Then $x(k) \le R$, $k \in \{0, \ldots, N+2\}$, and $x - \eta u_0 \in \Omega$ or

$$x(k) - \eta u_0(k) \ge 0, \quad k \in \{0, \ldots, N+2\}.$$

Because

$$\frac{\epsilon(x(k) - \eta u_0(k))}{R_1 + x(k) - \eta u_0(k)} \le \frac{\epsilon x(k)}{R_1 + x(k)}, \quad k \in \{0, \ldots, N+2\},$$

we get

$$\begin{aligned}
\beta\left(\eta u_0 + T(x - \eta u_0) + Sx\right) &= \beta\left(\eta u_0 - \frac{\epsilon(x - \eta u_0)}{R_1 + x - \eta u_0} + \frac{\epsilon x}{R_1 + x} + x + S_2 x + B_1\right) \\
&\ge \beta\left(x + \frac{B_1}{2}\right) \\
&> \beta(x) \\
&= R
\end{aligned}$$

and hence,

$$\beta\left(\eta u_0 + T(x - \eta u_0) + Sx\right) \ne R.$$

All conditions of $(\mathcal{A}_1)$ of Theorem 4.1 are then satisfied. Thus, we conclude that the BVP (4.7)-(4.8) has at least one solution $u^* \in \mathcal{P}$ such that $\frac{r}{A_1} \le \|u^*\| \le R$. This completes the proof.

□

Example

Let

$$\epsilon = B = A_1 = \frac{1}{10^{500}}, \quad R = 1, \quad B_1 = \frac{2}{10^{400}}, \quad r = \frac{1}{10^{600}}, \quad N = 5, \quad p = 2, \quad R_1 = 100.$$

Then

$$\begin{aligned}
A_1(N+3)\left((N+2)(N+1)B\left(1 + R^p\right) + R\right) &= \frac{1}{10^{500}} \cdot 8 \cdot \left(7 \cdot 6 \cdot \frac{1}{10^{500}}(1+1) + 1\right) \\
&< \frac{1}{10^{400}} = \frac{B_1}{2}
\end{aligned}$$

and

$$R_1 = 100 > r, \quad \frac{r}{A_1} = \frac{1}{10^{100}} < R,$$

and

$$A_1(\epsilon + r + 2B_1) = \frac{1}{10^{500}}\left(\frac{1}{10^{500}} + \frac{1}{10^{600}} + \frac{4}{10^{400}}\right) < \frac{1}{10^{600}} = r.$$

Thus, $(\mathcal{H}_2)$ holds. Now, by our main result, it follows that the BVP

$$\begin{aligned} \Delta^2 u(k) &= \frac{k}{10^{1000}(1+k+k^2)} + \frac{1}{10^{500}}(u(k))^2, \quad k \in \{0, \ldots, 5\}, \\ u(0) &= u(7) = 0 \end{aligned}$$

has at least one positive solution.

4.4 Concluding remarks

(1) In this chapter we have presented a new functional fixed point theorem on cones for the sum of two operators. The arguments are based upon recent fixed point index theory in cones of Banach spaces.

(2) By utilizing our approach, sufficient conditions for the existence of at least one positive solution are established for a non-autonomous second order difference equation.

(3) The nonlinearity f considered in the BVP (4.7)-(4.8) is non-autonomous and satisfies a general growth condition, while in [75] the nonlinear term must be autonomous with some conditions of monotonic type. Moreover, one can easily give an example for the constants ϵ, A_1, B, B_1, R, R_1, r which satisfy the condition $(\mathcal{H}_2)$.

(4) The functionals α and β considered in this work are more general than those in [75]. They are supposed to be only nonnegative and continuous, while in [75] the functionals α and β besides of being nonnegative and continuous were assumed concave and convex, respectively.

(5) For all the above reasons, our new topological approach developed in this chapter can be used to study other types of difference equations as well as dynamic equations.

Part III

Differential equations

5 New multiple fixed point theorems and application to Sturm-Liouville BVP

The results of this chapter are obtained by Bouchal and Mebarki in [22].

5.1 Introduction

In this chapter, we develop new multiple fixed point theorems for the sum of k-set contraction perturbed by an operator T such that $(I - T)$ is Lipschitz invertible on translate of cones. Therefore, existence criteria for at least three positive solutions for a singular generalized Sturm-Liouville multipoint boundary value problem are established, we also discussed the existence of countably many solutions. This study is carried out under conditions much weaker than those imposed in [94] and [93].

5.2 Multiple fixed point theorems for the sum of two operators

It is well-known that if D is bounded open subset of a Banach space E and A is a strict set contraction mapping defined on the closure of D and taking values in E, then the Leray-Schauder boundary condition:

$$Ax \neq \lambda x \text{ for all } x \in \partial D, \lambda > 1,$$

is sufficient to guarantee the existence of a fixed point for A. For the importance of this condition and its extensions in the study of nonlinear problems, we refer the reader to [40, 57]. In this work, we develop an extension of the Leray-Schauder boundary condition by considering a translate of cone $\mathcal{P}$ defined by $\mathcal{K}_w = \mathcal{P} + w = \{x + w : \ x \in \mathcal{P}\}$ for $w \in E$. First, we present our result for the class of strict set contractions. Next, we extend it for some class of k-set contractions perturbed by an operator T such that $(I - T)$ is Lipschitz invertible.

Lemma 5.1. *Let $\mathcal{K}_\omega$ be a translate of a cone $\mathcal{P}$ and $U \subset \mathcal{K}_\omega$ a bounded open subset with $\omega \in U$. Assume that $A : \overline{U} \to \mathcal{K}_\omega$ is a strict k-set contraction without fixed point on ∂U and there exists $\varepsilon > 0$ small enough such that*

$$Ax - \omega \neq \lambda(x - \omega) \ \ \textit{for all} \ \ x \in \partial U \ \ \textit{and} \ \ \lambda \geq 1 + \varepsilon. \tag{5.1}$$

Then the fixed point index $i\,(A, U, \mathcal{K}_\omega) = 1$.

Proof. Consider the homotopic deformation $H : [0,1] \times \overline{U} \to \mathcal{K}_\omega$ defined by

$$H(t,x) = \frac{t}{\varepsilon + 1}\,(Ax - \omega) + \omega.$$

The operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t,.)$ is a strict set contraction for each $t \in [0,1]$. In addition, $H(t,.)$ has no fixed point on ∂U. Otherwise, there would exist some $x_0 \in \partial U$ and $t_0 \in [0,1]$ such that $\frac{t_0}{\varepsilon+1}\,(Ax_0 - \omega) + \omega = x_0$, then

- If $t_0 = 0$, we get $x_0 = \omega$, contradicting $\omega \in U$.
- If $t_0 \in (0,1]$, we get $Ax_0 - \omega = \frac{1+\varepsilon}{t_0}(x_0 - \omega)$ with $\frac{1+\varepsilon}{t_0} \geq 1 + \varepsilon$, contradicting the assumption (5.1).

From the invariance under homotopy and the normalization properties of the index, we deduce

$$i\,(\frac{1}{\varepsilon + 1}\,A + \frac{\varepsilon}{\varepsilon + 1}\,\omega, U, \mathcal{K}_\omega) = i\,(\omega, U, \mathcal{K}_\omega) = 1.$$

Now, we show that

$$i\,(A, U, \mathcal{K}_\omega) = i\,(\frac{1}{\varepsilon + 1}\,A + \frac{\varepsilon}{\varepsilon + 1}\,\omega\,, U, \mathcal{K}_\omega).$$

Since A has no fixed point in ∂U and $(I-A)(\partial U)$ is a closed set (see [78, Lemma 1]), we get $0 \notin \overline{(I-A)(\partial U)}$.

Hence,

$$\gamma := dist(0,(I-A)(\partial U)) = \inf_{x\in\partial U} \|x - Ax\| > 0.$$

Let ε be sufficiently small so that $\|\frac{\varepsilon}{\varepsilon+1}(Ax-\omega)\| < \frac{\gamma}{2}$ and $\frac{\varepsilon+2}{\varepsilon+1}k < 1$. Hence

$$\left\|Ax - \left(\frac{1}{\varepsilon+1}Ax + \frac{\varepsilon}{\varepsilon+1}\omega\right)\right\| = \left\|\frac{\varepsilon}{\varepsilon+1}\Big(Ax-\omega\Big)\right\|,\ \forall\, x \in \partial U.$$

Define the convex deformation $G : [0,1]\times\overline{U} \to \mathcal{K}_\omega$ by

$$G(t,x) = tAx + (1-t)\left(\frac{1}{\varepsilon+1}\,Ax + \frac{\varepsilon}{\varepsilon+1}\,\omega\right).$$

The operator G is continuous and uniformly continuous in t for each x, and the mapping $G(t,.)$ is strict set contraction, with constant $\frac{\varepsilon+2}{\varepsilon+1}k$, for each $t\in[0,1]$. In addition, $G(t,.)$ has no fixed point on ∂U. In fact, for all $x\in\partial U$, we have

$$\begin{aligned}
\|x - G(t,x)\| &= \left\|x - tAx - (1-t)\left(\tfrac{1}{\varepsilon+1}\,Ax + \tfrac{\varepsilon}{\varepsilon+1}\,\omega\right)\right\| \\
&\geq \left\|x - \left(\tfrac{1}{\varepsilon+1}\,Ax + \tfrac{\varepsilon}{\varepsilon+1}\,\omega\right)\right\| - t\left\|Ax - \left(\tfrac{1}{\varepsilon+1}\,Ax + \tfrac{\varepsilon}{\varepsilon+1}\,\omega\right)\right\| \\
&= \left\|x - Ax + \tfrac{\varepsilon}{\varepsilon+1}\,Ax - \tfrac{\varepsilon}{\varepsilon+1}\,\omega\right\| - t\left\|\tfrac{\varepsilon}{\varepsilon+1}\Big(Ax-\omega\Big)\right\| \\
&\geq \|x - Ax\| - \left\|\tfrac{\varepsilon}{\varepsilon+1}\Big(Ax-\omega\Big)\right\| - t\left\|\tfrac{\varepsilon}{\varepsilon+1}\Big(Ax-\omega\Big)\right\| \\
&> \gamma - \tfrac{\gamma}{2} - \tfrac{\gamma}{2} = 0.
\end{aligned}$$

Then our claim follows from the invariance by homotopy property of the index.

□

Now, we extend the previous result to the case of a k-set contraction perturbed by an operator T such that $(I-T)$ is Lipschitz invertible.

Lemma 5.2. *Let $\mathcal{K}_\omega$ be a translate of a cone $\mathcal{P}$. Let Ω be a subset of $\mathcal{K}_\omega$ and U a bounded open subset of $\mathcal{K}_\omega$. Assume that $T : \Omega \to E$ be such that $(I - T)$ is Lipschitz invertible mapping with constant $\gamma > 0$, $S : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$ and $S(\overline{U}) \subset (I - T)(\Omega)$. Suppose that $T + S$ has no fixed point on $\partial U \cap \Omega$. Then we have the following results:*

(1) *If $\omega \in U$ and there exists $\varepsilon > 0$ small enough such that*

$$Sx \neq (I - T)(\lambda x + (1 - \lambda)\omega) \ \text{ for all } \ \lambda \geq 1 + \varepsilon, \ x \in \partial U \ \text{ and } \ \lambda x + (1 - \lambda)\omega \in \Omega,$$

then the fixed point index $i_* (T + S, U \cap \Omega, \mathcal{K}_\omega) = 1$.

(2) *If there exists $u_0 \in \mathcal{P}^*$ such that*

$$Sx \neq (I - T)(x - \lambda u_0), \ \text{ for all } \ \lambda > 0 \ \text{ and } \ x \in \partial U \cap (\Omega + \lambda u_0),$$

then the fixed point index $i_* (T + S, U \cap \Omega, \mathcal{K}_\omega) = 0$.

Proof. **(1)** The mapping $(I - T)^{-1}S : \overline{U} \to \mathcal{K}_\omega$ is a strict γk-set contraction without fixed point on $\partial U \cap \Omega$, and our hypothesis implies

$$(I - T)^{-1}Sx - \omega \neq \lambda(x - \omega) \quad \text{for} \quad \text{all} \quad x \in \partial U \quad \text{and} \quad \lambda \geq 1 + \varepsilon.$$

Then, our claim follows from (1.10) and Lemma 5.1.

(2) The mapping $(I - T)^{-1}S : \overline{U} \to \mathcal{K}_\omega$ is a strict γk-set contraction.

Assume by contradiction that $i_* (T + S, U \cap \Omega, \mathcal{K}_\omega) \neq 0$, then

$$i((I - T)^{-1}S, U, \mathcal{K}_\omega) \neq 0.$$

For each $r > 0$, define the homotopy:

$$H(t, x) = (I - T)^{-1}Sx + tru_0, \ \text{for } x \in \overline{U} \text{ and } t \in [0, 1].$$

The operator H is continuous and uniformly continuous in t for each x. Moreover, $H(t, .)$ is a strict γk-set contraction mapping for each t, and

$$H([0, 1] \times \overline{U}) = ((I - T)^{-1}S(\overline{U}) + tru_0) \subset \mathcal{K}_\omega.$$

In addition, $H(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial U$. Otherwise, there would exist some $(t_0, x_0) \in [0,1] \times \partial U$ such that $H(t_0,x_0) = x_0$, then

$$(I-T)^{-1}Sx_0 = x_0 - t_0 r u_0,$$

and so $x_0 - t_0 r u_0 \in \Omega$. Hence

$$Sx_0 = (I-T)(x_0 - t_0 r u_0),$$

for some $x_0 \in \partial U \cap (\Omega + t_0 r u_0)$, which contradict our assumption. By homotopy invariance property of the fixed point index, we deduce that

$$i((I-T)^{-1}S + ru_0, U, \mathcal{K}_\omega) = i((I-T)^{-1}S, U, \mathcal{K}_\omega) \neq 0.$$

Thus, from the existence property of the fixed point index, for each $r > 0$, there exists $x_r \in U$ such that

$$x_r - (I-T)^{-1}Sx_r = ru_0. \tag{5.2}$$

Letting $r \to +\infty$ the left hand side of (5.2) is bounded, while the right hand side is not, which is a contradiction. Therefore

$$i_*(T+S, U \cap \Omega, \mathcal{K}_\omega) = 0.$$

□

Remark 5.1. *(a). The result (1) in Lemma 5.2 is an extension of [34, Proposition 2.11], [43, Proposition 4.1] and [33, Proposition 4].*

(b). The result (2) in Lemma 5.2 and additional results concerning the computation of the fixed point index for the sum $T+S$ on translates of cones, are given in [43].

The following result prove the existence of at least three fixed points for the operator $T+S$ on translates of cones.

Theorem 5.1. *Let U_1, U_2 and U_3 three open bounded subsets of $\mathcal{K}_\omega$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $\omega \in U_1$,and Ω be a subset of $\mathcal{K}_\omega$. Assume that $T : \Omega \to E$ be such that $(I-T)$ is Lipschitz invertible mapping with constant $\gamma > 0$, $S : \overline{U}_3 \to E$ a k-set contraction with $0 \leq k < \gamma^{-1}$ and*

$S(\overline{U}_3) \subset (I-T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exist $u_0 \in \mathcal{P}^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:

(i) $Sx \neq (I-T)(\lambda x + (1-\lambda)\omega)$, *for all* $\lambda \geq 1+\varepsilon$, $x \in \partial U_1$ *and* $\lambda x + (1-\lambda)\omega \in \Omega$,

(ii) $Sx \neq (I-T)(x - \lambda u_0)$, *for all* $\lambda \geq 0$ *and* $x \in \partial U_2 \cap (\Omega + \lambda u_0)$,

(iii) $Sx \neq (I-T)(\lambda x + (1-\lambda)\omega)$, *for all* $\lambda \geq 1+\varepsilon$, $x \in \partial U_3$ *and* $\lambda x + (1-\lambda)\omega \in \Omega$,

Then $T+S$ has at least three nontrivial fixed points $x_1, x_2, x_3 \in \mathcal{K}_\omega$ such that

$$x_1 \in \partial U_1 \cap \Omega \text{ and } x_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega \text{ and } x_3 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega,$$

or

$$x_1 \in U_1 \cap \Omega \text{ and } x_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega \text{ and } x_3 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Proof. If $Sx = (I-T)x$ for $x \in \partial U_1 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_1 \cap \Omega$ of the operator $T+S$. Suppose that $Tx + Sx \neq x$ on $\partial U_1 \cap \Omega$. Without loss of generality, assume that $Tx + Sx \neq x$ on $\partial U_3 \cap \Omega$. By Lemma 5.2, we have

$$i_* (T+S, U_1 \cap \Omega, \mathcal{K}_\omega) = i_* (T+S, U_3 \cap \Omega, \mathcal{K}_\omega) = 1.$$

and

$$i_* (T+S, U_2 \cap \Omega, \mathcal{K}_\omega) = 0,$$

From the additivity property of the index i_*, we get

$$i_* (T+S, (U_2 \setminus \overline{U}_1) \cap \Omega, \mathcal{K}_\omega) = -1,$$

$$i_* (T+S, (U_3 \setminus \overline{U}_2) \cap \Omega, \mathcal{K}_\omega) = 1,$$

Consequently, by the existence property of the index i_*, $T+S$ has at least three fixed points such that $x_1 \in U_1 \cap \Omega$, $x_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega$ and $x_3 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$. □

Similarly, we can prove the following results, which are extensions of Theorem 5.1.

Theorem 5.2. *Let $U_1, U_2, \ldots, U_n$ be n $(n \geq 3)$ open bounded subsets of $\mathcal{K}_\omega$ such that $\overline{U}_1 \subset \overline{U}_2 \subset \cdots \subset U_n$ and $\omega \in U_1$ and Ω be a subset of $\mathcal{K}_\omega$. Assume that $T : \Omega \to E$ be such that $(I - T)$ is Lipschitz invertible mapping with constant $\gamma > 0$, $S : \overline{U}_n \to E$ is a ℓ-set contraction with $0 \leq \ell < \gamma^{-1}$ such that $T + S$ has no fixed point in $\partial U_{2k+1} \cap \Omega$ for $2k + 1 \in \{1, \ldots, n\}$ and $S(\overline{U}_n) \subset (I - T)(\Omega)$.*

Suppose that $(U_{i+1} \setminus \overline{U}_i) \cap \Omega \neq \emptyset$, for $i \in \{1, \ldots, n-1\}$, and there exist $u_0 \in \mathcal{P}^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:

(a) $Sx \neq (I - T)(\lambda x + (1 - \lambda)\omega)$, *for all* $\lambda \geq 1 + \varepsilon$, $x \in \partial U_{2k+1}$, *for* $2k + 1 \in \{1, \ldots, n\}$ *and* $(\lambda x + (1 - \lambda)\omega) \in \Omega$.

(b) $Sx \neq (I - T)(x - \lambda u_0)$, *for all* $\lambda \geq 0$ *and* $x \in \partial U_{2k} \cap (\Omega + \lambda u_0)$, *for* $2k \in \{2, \ldots, n\}$.

Then $T + S$ has n nontrivial fixed points $x_1, x_2, \ldots, x_n \in \mathcal{K}_\omega$ satisfying

$$x_1 \in U_1 \cap \Omega, \quad \textit{and} \quad x_i \in (U_i \setminus \overline{U}_{i-1}) \cap \Omega, \quad \textit{for} \quad i \in \{2, \ldots, n\}.$$

Theorem 5.3. *Let $U_1, U_2, \ldots, U_{n+1}$ be $n + 1$ $(n \geq 3)$ open bounded subsets of $\mathcal{K}_\omega$ such that $\overline{U}_1 \subset \overline{U}_2 \subset \cdots \subset U_{n+1}$ and $\omega \in U_1$. Let $T : \Omega \to E$ be such that $(I - T)$ is Lipschitz invertible mapping with constant $\gamma > 0$, $S : \overline{U}_{n+1} \to E$ is a k-set contraction with $0 \leq \ell < \gamma^{-1}$ such that $T + S$ has no fixed point in $\partial U_{2k} \cap \Omega$ for $2k \in \{2, \ldots, n + 1\}$ and $S(\overline{U}_{n+1}) \subset (I - T)(\Omega)$.*

Suppose that $(U_{i+1} \setminus \overline{U}_i) \cap \Omega \neq \emptyset$, for $i \in \{1, \ldots, n-1\}$, and there exist $u_0 \in \mathcal{P}^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:

(a) $Sx \neq (I-T)(x-\lambda u_0)$, *for all* $\lambda \geq 0$ *and* $x \in \partial U_{2k+1} \cap (\Omega + \lambda u_0)$, *for* $2k+1 \in \{1, \ldots, n+1\}$.

(b) $Sx \neq (I - T)(\lambda x + (1 - \lambda)\omega)$, *for all* $\lambda \geq 1 + \varepsilon$, $x \in \partial U_{2k}$, *for* $2k \in \{2, \ldots, n + 1\}$ *and* $(\lambda x + (1 - \lambda)\omega) \in \Omega$.

Then $T + S$ has n nontrivial fixed points $x_1, x_2, \ldots, x_n \in \mathcal{K}_\omega$ satisfying

$$x_i \in (U_{i+1} \setminus \overline{U}_i) \cap \Omega, \quad \textit{for} \quad i \in \{1, \ldots, n\}.$$

5.3 Application to ODE's

In this section, by using our new topological approach for the sum of two operators, developed in Section 5.2, we discuss the existence of multiple positive solutions for the following singular Sturm-Liouville multipoint boundary value problem (BVP for short) :

$$\begin{aligned} -u''(t) &= h(t)f(t,u(t),u'(t)), \quad 0<t<1, \\ au(0)-bu'(0) &= \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1)+du'(1) &= \sum_{i=1}^{m-2} b_i u(\xi_i), \end{aligned} \tag{5.3}$$

where $f \in \mathcal{C}([0,1]\times\mathbb{R}^*\times\mathbb{R},\mathbb{R})$, $h \in \mathcal{C}([0,1],\mathbb{R})$, $a,b,c,d \in [0,\infty)$, $0<\xi_1<\xi_2<...<\xi_{m-2}<1\,(m\geq 3)$, $a_i, b_i \in [0,\infty)$ are constants for $i=1,2,\dots,m-2$ and $\rho = ac+ad+bc>0$.
By a positive solution, it means a function $u \in \mathcal{C}^1([0,1])\cap\mathcal{C}^2((0,1))$ such that $u(t)\geq 0$ on $[0,1]$ not identically zero and u satisfies (5.3).

Multipoint boundary value problem theory has been developed rapidly over the past twenty years. Since the original work of Il'in and Moiseev [53] on the existence of solutions for a linear multipoint BVP, special attention has been paid to the study of multipoint BVP for nonlinear ordinary differential equations. Different approaches have been used to deal with such kind of problems: Leray-Schauder continuation theorem, fixed point theorems in cones, coincidence degree theory and the method of upper and lower solutions.
In [94], by using the fixed point theorem of Avery and Peterson, Zhang and Sun discussed the existence of three positive solutions to the problem (5.3) in the case where $h \in \mathcal{C}([0,1],[0,\infty))$.
In [93], by using the same approach, Zhang obtained a multiplicity result for this problem in the singular case (where h may be singular at $t=0$ and /or $t=1$).

5.3.1 Integral formulation

Let $x(t) = at + b$ and $y(t) = d + c(1-t)$ for $t \in [0,1]$ and denote

$$\Delta := \begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i x(\xi_i) & \rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i) \\ \\ \rho - \sum\limits_{i=1}^{m-2} b_i x(\xi_i) & -\sum\limits_{i=1}^{m-2} b_i y(\xi_i) \end{vmatrix}.$$

In [67], it is proved that, if $\Delta \neq 0$, then the problem (5.3) is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t,s)h(s)f(s,u(s),u'(s))\,ds + \mathcal{A}(hf)\,x(t) + \mathcal{B}(hf)\,y(t), \quad t \in [0,1], \tag{5.4}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} (d + c(1-t))(as+b), & 0 \le s \le t \le 1, \\ (at+b)(d+c(1-s)), & 0 \le t \le s \le 1, \end{cases} \tag{5.5}$$

$$\mathcal{A}(v) := \frac{1}{\Delta}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i \int\limits_0^1 G(\xi_i,s)v(s)\,ds & \rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i) \\ \\ \sum\limits_{i=1}^{m-2} b_i \int\limits_0^1 G(\xi_i,s)v(s)\,ds & -\sum\limits_{i=1}^{m-2} b_i y(\xi_i) \end{vmatrix},$$

$$\mathcal{B}(v) := \frac{1}{\Delta}\begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i x(\xi_i) & \sum\limits_{i=1}^{m-2} a_i \int\limits_0^1 G(\xi_i,s)v(s)\,ds \\ \\ \rho - \sum\limits_{i=1}^{m-2} b_i x(\xi_i) & \sum\limits_{i=1}^{m-2} b_i \int\limits_0^1 G(\xi_i,s)v(s)\,ds \end{vmatrix}.$$

Let E be the Banach space $\mathcal{C}^1([0,1])$ endowed with the norm

$$\|u\| = \max\left(\max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |u'(t)|\right).$$

For $u \in E$, we define the operators

$$\begin{aligned} Fu(t) &= \int_0^1 G(t,s)h(s)f(s,u(s),u'(s))\,ds + \mathcal{A}(hf)x(t) + \mathcal{B}(hf)y(t), \\ S_1u(t) &= Fu(t) - u(t), \\ S_2u(t) &= \int_0^t (t-s)^2\, g(s)S_1u(s)\,ds, \quad t \in [0,1],\ g \in \mathcal{C}([0,1],(0,\infty)). \end{aligned}$$

By (5.4), it follows that if $u \in E$ satisfies the equation $S_1u = 0$, then it is a solution to the problem (5.3).

5.3.2 Assumptions

We first enunciate the common assumptions that we will use in order to prove our main results. Further assumptions will be assumed in each existence criteria.

$(\mathcal{H}_1)$ $f \in \mathcal{C}([0,1] \times [0,\infty) \times (-\infty,\infty), (-\infty,\infty))$,

$$|f(t,u,v)| \leq k_1|u|^{p_1} + k_2|v|^{p_2} + k_3, \quad t \in [0,1], \; u,v \in \mathbb{R},$$

k_1, k_2, k_3, p_1, p_2 are positive constants.

$(\mathcal{H}_2)$ $h \in \mathcal{C}((0,1),\mathbb{R})$ may be singular at $t = 0$ and/or $t = 1$ and $\int\limits_0^1 G(s,s)h(s)\,ds < \infty$, where G is given by (5.5).

$(\mathcal{H}_3)$ $\Delta < 0,\; \rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i) > 0,\; \rho - \sum\limits_{i=1}^{m-2} b_i x(\xi_i) > 0.$

5.3.3 Auxiliary results

Lemma 5.3. *Suppose that $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ hold. Let L a real constant and $u \in E$ satisfies the equation*

$$S_2 u(t) + 2L = 0, \quad t \in [0,1]. \tag{5.6}$$

Then u is a solution to the problem (5.3).

Proof. We differentiate the integral equation (5.6) three times with respect to t and we get

$$g(t)\, S_1 u(t) = 0, \quad t \in [0,1],$$

whereupon

$$S_1\, u(t) = 0, \quad t \in [0,1].$$

This completes the proof. □

Set

$$\mathcal{M} := \int_0^1 G(s,s)|h(s)|\,ds,$$

$$\mathbb{A} := \frac{1}{|\Delta|}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i & \rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i) \\ \\ -\sum\limits_{i=1}^{m-2} b_i & \sum\limits_{i=1}^{m-2} b_i y(\xi_i) \end{vmatrix},$$

$$\mathbb{B} := \frac{1}{|\Delta|}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i x(\xi_i) & -\sum\limits_{i=1}^{m-2} a_i \\ \\ \rho - \sum\limits_{i=1}^{m-2} b_i x(\xi_i) & \sum\limits_{i=1}^{m-2} b_i \end{vmatrix}.$$

Fix $B > 0$ arbitrarily.

Lemma 5.4. *Suppose that $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ hold. For any $u \in E$ with $\|u\| \leq B$, we have*

$$|Fu(t)| \leq \mathcal{M}\left(1 + (a+b)\mathbb{A} + (c+d)\mathbb{B}\right)\left(k_1 B^{p_1} + k_2 B^{p_2} + k_3\right),\ t \in [0,1].$$

Proof. We have

$$\begin{aligned}
|\mathcal{A}(hf)| \;&\leq\; \tfrac{1}{|\Delta|}\Bigg(\left(\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)|h(s)||f(s,u(s),u'(s))|\,ds\right)\left(\sum_{i=1}^{m-2} b_i y(\xi_i)\right) \\
&\quad + \left(\sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)|h(s)||f(s,u(s),u'(s))|\,ds\right)\left(\rho - \sum_{i=1}^{m-2} a_i y(\xi_i)\right)\Bigg) \\
&\leq\; \tfrac{1}{|\Delta|}\Bigg(\left(\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)|h(s)|\,ds\right)\left(\sum_{i=1}^{m-2} b_i y(\xi_i)\right) \\
&\quad + \left(\sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)|h(s)|\,ds\right)\left(\rho - \sum_{i=1}^{m-2} a_i y(\xi_i)\right)\Bigg)\left(k_1|u|^{p_1} + k_2|u|^{p_2} + k_3\right) \\
&\leq\; \tfrac{1}{|\Delta|}\Bigg(\left(\sum_{i=1}^{m-2} a_i \int_0^1 G(s,s)|h(s)|\,ds\right)\left(\sum_{i=1}^{m-2} b_i y(\xi_i)\right) \\
&\quad + \left(\sum_{i=1}^{m-2} b_i \int_0^1 G(s,s)|h(s)|\,ds\right)\left(\rho - \sum_{i=1}^{m-2} a_i y(\xi_i)\right)\Bigg)\left(k_1\|u\|^{p_1} + k_2\|u\|^{p_2} + k_3\right) \\
&\leq\; \mathcal{M}\mathbb{A}\left(k_1 B^{p_1} + k_2 B^{p_2} + k_3\right).
\end{aligned}$$

Similarly, we obtain

$$|\mathcal{B}(hf)| \leq \mathcal{M}\mathbb{B}\left(k_1 B^{p_1} + k_2 B^{p_2} + k_3\right).$$

Then

$$
\begin{aligned}
|Fu(t)| &\leq \int_0^1 G(t,s)|h(s)||f(s,u(s),u'(s))|\,ds + |\mathcal{A}(hf)|\,x(t) + |\mathcal{B}(hf)|\,y(t) \\
&\leq \int_0^1 G(t,s)|h(s)|\left(k_1|u(s)|^{p_1} + k_2|u'(s)|^{p_2} + k_3\right)\,ds \\
&\quad + \mathcal{M}\mathbb{A}((k_1\|u\|^{p_1} + k_2\|u\|^{p_2} + k_3))x(t) \\
&\quad + \mathcal{M}\mathbb{B}((k_1\|u\|^{p_1} + k_2\|u\|^{p_2} + k_3))y(t) \\
&\leq \mathcal{M}\left(1 + (a+b)\mathbb{A} + (c+d)\mathbb{B}\right)\left(k_1B^{p_1} + k_2B^{p_2} + k_3\right), \quad t \in [0,1].
\end{aligned}
$$

This completes the proof. □

Suppose

$(\mathcal{H}_4)$ $g \in \mathcal{C}([0,1],(0,\infty))$ be such that

$$\int_0^1 ((1-s)^2 + 2(1-s) + 2)\,g(s)\,ds \leq A_1,$$

for some constant $A_1 > 0$.

Lemma 5.5. *Suppose that $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ hold. Let $u \in E$ be such that $\|u\| \leq B$. Then*

$$\|S_2u\| \leq A_1(\mathcal{M}\left(1 + (a+b)\mathbb{A} + (c+d)\mathbb{B}\right)\left(k_1B^{p_1} + k_2B^{p_2} + k_3\right) + B),$$

$$|(S_2u)''(t)| \leq A_1(\mathcal{M}\left(1 + (a+b)\mathbb{A} + (c+d)\mathbb{B}\right)\left(k_1B^{p_1} + k_2B^{p_2} + k_3\right) + B), t \in [0,1].$$

Proof. Using Lemma 5.4, we arrive at

$$
\begin{aligned}
|S_2u(t)| &= \left|\int_0^t (t-s)^2 g(s) S_1u(s)\,ds\right| \\
&\leq \int_0^t (t-s)^2 g(s)|S_1u(s)|\,ds \\
&\leq \int_0^t (t-s)^2 g(s)|Fu(s) - u(s)|\,ds \\
&\leq \int_0^1 (1-s)^2 g(s)(|Fu(s)| + |u(s)|)\,ds \\
&\leq (\mathcal{M}\left(1 + (a+b)\mathbb{A} + (c+d)\mathbb{B}\right)\left(k_1B^{p_1} + k_2B^{p_2} + k_3\right) + B)\int_0^1 (1-s)^2 g(s)\,ds \\
&\leq A_1(\mathcal{M}\left(1 + (a+b)\mathbb{A} + (c+d)\mathbb{B}\right)\left(k_1B^{p_1} + k_2B^{p_2} + k_3\right) + B), \quad t \in [0,1],
\end{aligned}
$$

and

$$\begin{aligned}
|(S_2u)'(t)| &= 2\left|\int_0^t (t-s)g(s)S_1u(s)\,ds\right| \\
&\le 2\int_0^t (t-s)g(s)|S_1u(s)|\,ds \\
&\le 2\int_0^t (t-s)g(s)|Fu(s)-u(s)|\,ds \\
&\le 2(\mathcal{M}\left(1+(a+b)\mathbb{A}+(c+d)\mathbb{B}\right)\left(k_1B^{p_1}+k_2B^{p_2}+k_3\right)+B)\int_0^1 (1-s)g(s)\,ds \\
&\le A_1(\mathcal{M}\left(1+(a+b)\mathbb{A}+(c+d)\mathbb{B}\right)\left(k_1B^{p_1}+k_2B^{p_2}+k_3\right)+B), \quad t\in[0,1],
\end{aligned}$$

Hence,

$$\|S_2u\| \le A_1(\mathcal{M}\left(1+(a+b)\mathbb{A}+(c+d)\mathbb{B}\right)\left(k_1B^{p_1}+k_2B^{p_2}+k_3\right)+B).$$

and

$$\begin{aligned}
|(S_2u)''(t)| &= \left|2\int_0^t g(s)S_1u(s)\,ds\right| \\
&\le 2\int_0^t g(s)|S_1u(s)|\,ds \\
&\le 2(\mathcal{M}\left(1+(a+b)\mathbb{A}+(c+d)\mathbb{B}\right)\left(k_1B^{p_1}+k_2B^{p_2}+k_3\right)+B)\int_0^1 g(s)\,ds \\
&\le A_1(\mathcal{M}\left(1+(a+b)\mathbb{A}+(c+d)\mathbb{B}\right)\left(k_1B^{p_1}+k_2B^{p_2}+k_3\right)+B), \quad t\in[0,1].
\end{aligned}$$

This completes the proof. □

5.3.4 Main results

In the sequel, suppose that the constant A_1 which appears in $(\mathcal{H}_4)$ satisfies the following inequality:

$$A_1\left(\mathcal{M}\left(1+(a+b)\mathbb{A}+(c+d)\mathbb{B}\right)\left(k_1R_1^{p_1}+k_2R_1^{p_2}+k_3\right)+R_1\right) < 2L_1, \tag{5.7}$$

where L_1, R_1 are such that $r_1 < L_1 < R_1$ with r_1 a positive constant.

The first main existence criteria is the following:

Theorem 5.4. *If the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ and the inequality (5.7) are satisfied, the problem (5.3) has at least three positive solutions u_1, u_2, $u_3 \in \mathcal{C}^1([0,1]) \cap \mathcal{C}^2((0,1))$ that satisfy*

$$0 \le \max\{\max_{t\in[0,1]} |u_1(t)|, \max_{t\in[0,1]} |u_1'(t)|\} \le r_1,$$

$$r_1 < \max\{\max_{t\in[0,1]} |u_2(t)|, \max_{t\in[0,1]} |u_2'(t)|\} < L_1,$$

$$L_1 < \max\{\max_{t\in[0,1]} |u_3(t)|, \max_{t\in[0,1]} |u_3'(t)|\} \leq R_1.$$

Proof. Let

$$\mathcal{P} = \{u \in E : u \geq 0 \quad \text{on} \quad [0,1]\}.$$

For $u \in \mathcal{P}$ let us define the operators T and S as follows:

$$\begin{aligned} Tu(t) &= (1+\mu\varepsilon)u(t) - \varepsilon L_1, \\ Su(t) &= -\varepsilon S_2 u(t) - \mu\varepsilon u(t) - \varepsilon L_1,\ t \in [0,1], \end{aligned}$$

where μ is a large enough positive constant and $\varepsilon \geq \frac{4}{\mu}\frac{L_1}{r_1}$. Note that any fixed point $u \in \mathcal{P}$ of the operator $T+S$ is a solution to the problem (5.3). Define

$$U_1 = \mathcal{P}_{r_1} = \{u \in \mathcal{P} : \|u\| < r_1\},$$

$$U_2 = \mathcal{P}_{L_1} = \{u \in \mathcal{P} : \|u\| < L_1\},$$

$$U_3 = \mathcal{P}_{R_1} = \{u \in \mathcal{P} : \|u\| < R_1\},$$

$$\varrho = \frac{1}{\mu}\ (A_1(\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1 R_1^{p_1} + k_2 R_1^{p_2} + k_3) + R_1] + 2L_2 + \mu R_1)$$

$$\Omega = \overline{P_\varrho} = \{v \in \mathcal{P} : \|v\| \leq \varrho\}.$$

1. For $u_1, u_2 \in \Omega$, we have

$$\|Tu_1 - Tu_2\| = (1+\mu\varepsilon)\|u_1 - u_2\|,$$

   then T is an expansive operator with constant $1+\mu\varepsilon$.

   So, $(I-T) : E \to E$ is Lipschitz invertible with constant $\frac{1}{\mu\varepsilon}$.

2. As in [93, Lemma 2.5], by applying Ascoli-Arzelà compactness criterion, we can prove that the operator S is completely continuous then S is 0-set contraction.

3. We prove that $S(\overline{P_{R_1}}) \subset (I-T)(\Omega)$. Let $u \in \overline{P_{R_1}}$ be arbitrarily chosen. Set

$$v = \frac{S_2 u + 2L_1 + \mu u}{\mu}.$$

It is clear that $v \geq 0$ and

$$\begin{aligned}
\|v\| &= \|\tfrac{1}{\mu}(S_2u + 2L_1 + \mu u)\| \\
&\leq \tfrac{1}{\mu}(\|S_2u\| + 2L_1 + \mu\|u\|) \\
&\leq \tfrac{1}{\mu}\left(A_1(\mathcal{M}\left(1 + (a+b)\mathbb{A} + (c+d)\mathbb{B}\right)\left(k_1R_1^{p_1} + k_2R_1^{p_2} + k_3\right) + R_1) + 2L_1 + \mu R_1\right) \\
&= \varrho.
\end{aligned}$$

Therefore $v \in \Omega$ and

$$\begin{aligned}
(I-T)v &= -\epsilon(\mu v - L_1) \\
&= -\epsilon(\mu(\tfrac{S_2u+2L_1+\mu u}{\mu}) - L_1) \\
&= -\epsilon(S_2u + \mu u + L_1) \\
&= Su.
\end{aligned}$$

Thus, $S(\overline{P_{R_1}}) \subset (I-T)(\Omega)$.

4. Assume that there exist $\frac{\varrho}{r_1} \geq \lambda_1 \geq \varepsilon + 1$ and $x_1 \in \partial\mathcal{P}_{r_1}$ ($\lambda_1 x_1 \in \Omega$ leads to $\lambda_1 \leq \frac{\varrho}{\|x_1\|}$) such that

$$Sx_1 = (I-T)(\lambda_1 x_1).$$

Then

$$-\varepsilon S_2x_1 - \mu\varepsilon x_1 - \varepsilon L_1 = -\varepsilon\mu\lambda_1 x_1 + \varepsilon L_1,$$

or equivalently

$$S_2x_1 = \mu(\lambda_1 - 1)x_1 - 2L_1.$$

So

$$\begin{aligned}
\|S_2x_1\| &= \|\mu(\lambda_1-1)x_1 - 2L_1\| \\
&\geq \mu(\lambda_1-1)\|x_1\| - 2L_1 \\
&\geq \mu(\lambda_1-1)r_1 - 2L_1 \\
&\geq \mu\varepsilon r_1 - 2L_1 \\
&\geq 2L_1.
\end{aligned}$$

Hence, a contradiction with one of the results of Lemma 5.5 and (5.7).

5. Assume that for any $u_0 \in \mathcal{P}^*$, there exist $\lambda_0 \geq 0$ and $x_0 \in \partial P_{L_1} \cap (\Omega + \lambda_0 u_0)$ such that

$$Sx_0 = (I-T)(x_0 - \lambda_0 u_0).$$

Then

$$-\varepsilon(S_2x_0 + \mu x_0 + L_1) = -\varepsilon(\mu(x_0 - \lambda_0 u_0) - L_1),$$

or equivalently

$$S_2x_0 = -(\lambda_0\mu u_0 + 2L_1).$$

So

$$\|S_2x_0\| = \|\lambda_0\mu u_0 + 2L_1\| \geq 2L_1,$$

which is a contradiction.

6. Assume that there exist $\frac{\varrho}{R_1} \geq \lambda_2 \geq \varepsilon + 1$ and $x_2 \in \partial\mathcal{P}_{R_1}$ such that

$$Sx_2 = (I - T)(\lambda_2 x_2).$$

Then

$$S_2x_2 = (\lambda_2 - 1)\mu x_2 - 2L_1.$$

So

$$\begin{aligned}
\|S_2x_2\| &= \|\mu(\lambda_1 - 1)x_2 - 2L_1\| \\
&\geq \mu(\lambda_2 - 1)R_1 - 2L_1 \\
&\geq \mu\varepsilon R_1 - 2L_1 \\
&\geq \mu\varepsilon r_1 - 2L_1 \\
&\geq 2L_1,
\end{aligned}$$

which is a contradiction.

Therefore all conditions of Theorem 5.1 hold for $w = 0$. Hence, the problem (5.3) has at least three solutions u_1, u_2 and u_3 in $\mathcal{P}$ so that

$$0 \leq \|u_1\| < r_1 < \|u_2\| < L_1 < \|u_3\| \leq R_1.$$

□

Now, we discuss the existence of countably many positive solutions for the problem (5.3). If we replace the inequality (5.7) by the following one

$$A_1\left(\mathcal{M}\left(1 + (a+b)\mathbb{A} + (c+d)\mathbb{B}\right)\left(k_1R_n^{p_1} + k_2R_n^{p_2} + k_3\right) + R_n\right) < 2L_1, \tag{5.8}$$

where $n \in \mathbb{N}^*$ fixed and for $i \in \{1, \ldots, n\}$, $L_i, R_i \in (0, \infty)$ are such that $r_i < L_i < R_i$ with $r_i > R_{i-1}, i \geq 2$, and by using similar arguments as in the proof of Theorem 5.4, we can prove the following generalized existence criteria:

Theorem 5.5. *If the assumptions* $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ *and the inequality* (5.8) *are satisfied, the problem* (5.3) *has at least* $2n+1$ *positive solutions* $u_k \in \mathcal{C}^1([0,1]) \cap \mathcal{C}^2((0,1))$, $k \in \{1, \ldots, 2n+1\}$ *that satisfy*

$$0 \leq \|u_1\| < r_1 < \|u_2\| < L_1 < \|u_3\| \leq R_1,$$

$$r_k \leq \|u_{2k}\| < L_k < \|u_{2k+1}\| \leq R_k, \text{ for } k \in \{2, \ldots, n\}.$$

Proof. In this case for $i \in \{1, \ldots, n\}$, we consider

$$U_1^{(i)} = \mathcal{P}_{r_i} = \{u \in \mathcal{P} : \|u\| < r_i\},$$

$$U_2^{(i)} = \mathcal{P}_{L_i} = \{u \in \mathcal{P} : \|u\| < L_i\},$$

$$U_3^{(i)} = \mathcal{P}_{R_i} = \{u \in \mathcal{P} : \|u\| < R_i\},$$

$$\varrho = \frac{1}{\mu}(A_1(\mathcal{M}\,(1 + (a+b)\mathbb{A} + (c+d)\mathbb{B})\,(k_1 R_n^{p_1} + k_2 R_n^{p_2} + k_3) + R_n) + 2L_1 + \mu R_n)$$

$$\Omega = \overline{\mathcal{P}_\varrho} = \{u \in \mathcal{P} : \|u\| \leq \varrho\}.$$

□

5.3.5 Examples

Let $m = 4$,

$$a = 4, \quad b = 2, \quad c = 4, \quad d = 2,$$
$$a_1 = \tfrac{1}{4}, \quad a_2 = \tfrac{1}{2}, \quad b_1 = \tfrac{1}{3}, \quad b_2 = \tfrac{1}{2},$$
$$\xi_1 = \tfrac{1}{4}, \quad \xi_2 = \tfrac{1}{2},$$

We consider the following BVP

$$\begin{aligned} -u''(t) &= h(t)f(t, u(t), u'(t)), \quad 0 < t < 1, \\ 4u(0) - 2u'(0) &= \tfrac{1}{4}u(\tfrac{1}{4}) + \tfrac{1}{2}u(\tfrac{1}{2}), \\ 4u(1) + 2u'(1) &= \tfrac{1}{3}u(\tfrac{1}{4}) + \tfrac{1}{2}u(\tfrac{1}{2}). \end{aligned} \tag{5.9}$$

Where

$$h(t) = \frac{1}{\sqrt{t}} + \frac{1}{\sqrt{1-t}}, \quad t \in (0,1),$$

$$f(t,y,z) = \frac{1}{10^2}\,t + \frac{1}{10^4}\,y + \frac{1}{10^4}\,z^{\frac{1}{5}}, \qquad t \in [0,1],\ y \in [0,\infty),\ z \in (-\infty,\infty).$$

Let also,

$$r_1 = 1, \quad L_1 = 15, \quad R_1 = 20,$$

$$r_2 = 25, \quad L_2 = 35, \quad R_2 = 40,$$

$$p_1 = \tfrac{1}{2}, \quad p_2 = \tfrac{1}{2}, \quad A_1 = \tfrac{1}{2},$$

$$k_1 = \tfrac{1}{3}, \quad k_2 = \tfrac{1}{3}, \quad k_3 = \tfrac{1}{3},$$

By some calculations, we have

$$\rho = 32, \quad \Delta = -823.66,$$

and the conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ hold.

We have,

$$G(s,s) = -\frac{1}{32}(4s+2)(4s-6).$$

Let

$$\mathcal{M} = \int_0^1 G(s,s)\,h(s)\,ds = \frac{53}{30}.$$

Then

$$A_1((\mathcal{M} + (a+b)\mathbb{A} + (c+d)\mathbb{B})\,(k_1R_2^{p_1} + k_2R_2^{p_2} + k_3) + R_2) = 25,4559 < 2L_1.$$

Let $g(s) = \frac{s+1}{10}$, $s \in [0,1]$. Then

$$\int_0^1 ((1-s)^2 + 2(1-s) + 2)\,g(s)\,ds = \frac{1}{10}\int_0^1 ((1-s)^2 + 2(1-s) + 2)\,s\,ds = \frac{19}{40} = 0,475 \le A_1.$$

Then all assumptions of Theorem 5.5 for $n = 2$. Hence, the problem (5.9) has at least five positive solutions u_1, u_2, u_3, u_4, u_5 such that

$$0 \le \|u_1\| \le 1,$$

$$1 < \|u_2\| < 10,$$

$$10 < \|u_3\| \leq 20.$$

$$25 \leq \|u_4\| < 35,$$

$$35 < \|u_5\| \leq 40.$$

5.4 Comparison and concluding remarks

In this section we compare the results obtained in this work with those obtained by Zhang-Sun [94] and Zhang [93].

(1) In this work the nonlinear term f takes values on $\mathbb{R}$ and it is involved with the first order derivative, in addition f satisfies a general growth condition. The nonlinearity considered in [94] and [93] takes values in $[0, \infty)$ and supposed piecewise bounded.

(2) The problem studied here is endowed with a singular term given by h which takes values on $\mathbb{R}$ and the integral of h on $(0, 1)$ do not have to be finite as in [93], it is sufficient that $\int_0^1 G(s, s)h(s)\, ds < \infty$.

(3) The conditions $a > \sum\limits_{i=1}^{m-2} a_i$, $c > \sum\limits_{i=1}^{m-2} b_i$ in both [94] and [93] are not of interest in our work.

(4) In this work, sufficient conditions for the existence of countably many positive solutions for the problem (5.3) are established. However, in [94] and [93] the authors have only discussed the existence of three positive solutions.

(5) Our approach has been applied to prove the existence of finite multiple positive solutions as well as the existence of a countable many positive solutions for the problem (5.3), and it can be used to study the existence of multiple solutions for other classes of differential equations covered by various types of boundary value problems.

6 Fixed point theorem on functional intervals for sum of two operators and application to ODEs

The results of this chapter are obtained by Bouchal and Mebarki in [21].

6.1 Introduction

This chapter is a part of generalization of some results in fixed point theory on cones of Banach spaces for the sum of two operators. More precisely, we are interested in the theorems of functional types and their applications in the study of boundary value problems. Note that, Leggett and Williams [63] were the originators of this class of fixed point theorems. Since then, the literature has had a significant number of functional fixed point theorems developed promptly in different directions, especially those due to Avery et al. [13, 10, 7, 6, 12]. From a mathematical point of view, when functionals are used in applications instead of norms, we get more freedom and flexibility.

Recently, in [34] a new direction of research in the theory of fixed point in ordered Banach spaces for the sum of two operators was opened. Then, several fixed point theorems, including Leggett-Williams theorems type in cones, have been developed (see [16, 19, 41, 42, 43, 72]). These theorems have been applied to discuss the existence of positive solutions for various types of boundary and/or initial value problems (see [39, 41, 43]).

In [11], Avery et al. have developed an extension of the compression-expansion fixed point theorem of functional type by generalizing the underlaying set using functional-type interval which are sets of the form

$$\mathcal{A}(\beta, b, \alpha, a) = \{x \in \mathcal{A} : \ a < \alpha(x) \text{ and } \beta(x) < b\},$$

where $\mathcal{A}$ is an open subset of a cone $\mathcal{P}$. Motivated by this work, we improve this result by considering the sum of two operators $T + F$, where $I - T$ is Lipschitz invertible and F a k-set contraction. The main tool used is a recent fixed point index theory in cones for this class of mappings developed by Mebarki et al. in [34, 43].

6.2 Fixed point theorem on functional intervals

In the sequel, we will establish an extension of [11, Theorem 3.1] which guarantees the existence of at least one nontrivial positive solution to some equations of the form $Tx + Fx = x$ posed on cones of a Banach space.

Definition 6.1. *A map α is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space E if $\alpha : \mathcal{P} \to [0, \infty)$ is continuous and*

$$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y),$$

for all $x, y \in \mathcal{P}$ and $t \in [0, 1]$.

A map β is said to be a nonnegative continuous convex functional on a cone $\mathcal{P}$ of a real Banach space E if $\beta : \mathcal{P} \to [0, \infty)$ is continuous and

$$\beta(tx + (1 - t)y) \leq t\beta(x) + (1 - t)\beta(y),$$

for all $x, y \in \mathcal{P}$ and $t \in [0, 1]$.

Let $\mathcal{A}$ be a relatively open subset of a cone $\mathcal{P}$, α be a nonnegative continuous concave functional on $\mathcal{P}$; β be a nonnegative continuous convex functional on $\mathcal{P}$ and let a, b be two positive real numbers, then the set

$$\mathcal{A}(\beta, b, \alpha, a) = \{x \in \mathcal{A} : \ a < \alpha(x) \text{ and } \beta(x) < b\},$$

is an interval of functional type.

Note that $\mathcal{A}(\beta, b, \alpha, a)$ is a subset of $\mathcal{P}(\beta, b, \alpha, a)$ defined by:

$$\mathcal{P}(\beta, b, \alpha, a) = \{x \in \mathcal{P} : \ a < \alpha(x) \text{ and } \beta(x) < b\}.$$

Theorem 6.1. *Let E be a Banach space; $\mathcal{P} \subset E$ be a cone, $\mathcal{A}$ be a relatively open subset of $\mathcal{P}$, α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$ and β and θ are nonnegative continuous convex functionals on $\mathcal{P}$.*

Let $T : \Omega \subset \mathcal{P} \to E$ be a mapping such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$ and $F : \mathcal{P} \to E$ be a k-set contraction mapping with $0 \leq k < \gamma^{-1}$.

Assume that there exist four nonnegative numbers a, b, c and d, and $\omega_0 \in \mathcal{A}(\beta, b, \alpha, a) \cap \mathcal{A}(\theta, c, \psi, d)$ such that

$$(I - T)^{-1}\omega_0 \in \mathcal{A}(\beta, b, \alpha, a), \tag{6.1}$$

$$\lambda F(\mathcal{A}(\beta, b, \alpha, a)) + (1 - \lambda)\omega_0 \subset (I - T)(\Omega), \ \textit{for all } \lambda \in [0, 1], \tag{6.2}$$

$(\mathcal{H}_1)$ *$\mathcal{A}(\beta, b, \alpha, a)$ is bounded, and $\partial\mathcal{A} \cap \overline{\mathcal{A}(\beta, b, \alpha, a)} = \emptyset$;*

$(\mathcal{H}_2)$ *if $x \in \mathcal{P}$ with $\alpha(x) = a$ and either $\theta(x) \leq c$ or $\theta(Tx + Fx) > c$, then $\alpha(Tx + Fx) > a$;*

$(\mathcal{H}_3)$ *if $x \in \mathcal{P}$ with $\beta(x) = b$ and either $\psi(Tx + Fx) < d$ or $\psi(x) \geq d$, then $\beta(Tx + Fx) < b$;*

$(\mathcal{H}_4)$ *if $x \in \mathcal{P}$ with $\alpha(x) = a$, then $\alpha(Tx + \omega_0) > a$ and $\theta(Tx + \omega_0) \leq c$;*

$(\mathcal{H}_5)$ *if $x \in \mathcal{P}$ with $\beta(x) = b$, then $\beta(Tx + \omega_0) < b$ and $\psi(Tx + \omega_0) \geq d$;*

then $T + F$ has at least one fixed point $x^* \in \overline{\mathcal{A}(\beta, b, \alpha, a)}$.

***Proof*.** **Claim 1:** $Tx + Fx \neq x$ for all $x \in \partial\mathcal{A}(\beta, b, \alpha, a)$.

The functional interval $\mathcal{A}(\beta, b, \alpha, a) = \mathcal{A} \cap \mathcal{P}(\beta, b, \alpha, a)$, then by the condition $(\mathcal{H}_1)$,

$$\partial\mathcal{A}(\beta, b, \alpha, a) = \overline{\mathcal{A}} \cap \partial\mathcal{P}(\beta, b, \alpha, a).$$

Suppose that there exist $z_0 \in \partial\mathcal{A}(\beta, b, \alpha, a)$ such that $Tz_0 + Fz_0 = z_0$. Since $z_0 \in \partial\mathcal{P}(\beta, b, \alpha, a)$ so either $\beta(z_0) = b$ or $\alpha(z_0) = a$.

Case 1: $\beta(z_0) = b$.

If $\psi(Tz_0 + Fz_0) < d$ or $\psi(z_0) = \psi(Tz_0 + Fz_0) \geq d$, then by the condition $(\mathcal{H}_3)$,

$$\beta(Tz_0 + Fz_0) < b.$$

Hence we have that $Tz_0 + Fz_0 \neq z_0$.

Case 2: $\alpha(z_0) = a$.

If $\theta(z_0) = \theta(Tz_0 + Fz_0) \leq c$ or $\theta(Tz_0 + Fz_0) > c$, then by the condition $(\mathcal{H}_2)$,

$$\alpha(Tz_0 + Fz_0) > a.$$

Hence we have that $Tz_0 + Fz_0 \neq z_0$.

Therefore, $T + F$ does not have any fixed points on $\partial\mathcal{A}(\beta, b, \alpha, a)$.

Claim 2: Let $H : [0, 1] \times \overline{\mathcal{A}(\beta, b, \alpha, a)} \to E$ be defined by

$$H(t, x) = tFx + (1 - t)\omega_0.$$

We have

(i) $H : [0, 1] \times \overline{\mathcal{A}(\beta, b, \alpha, a)} \to E$ is continuous and $H(t, x)$ is uniformly continuous in t with respect to $x \in \overline{\mathcal{A}(\beta, b, \alpha, a)}$,

(ii) $H([0, 1] \times \mathcal{A}(\beta, b, \alpha, a)) \subset (I - T)(\Omega)$,

(iii) $H(t, .) : \overline{\mathcal{A}(\beta, b, \alpha, a)} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$ for all $t \in [0, 1]$,

(iv) $Tx + H(t, x) \neq x$, for all $t \in [0, 1]$ and $x \in \partial\mathcal{A}(\beta, b, \alpha, a) \cap \Omega$.

Properties (i), (ii) and (iii) follow directly from the definition of H and the conditions on F and T . We only check (iv).

Suppose the contrary, that is, there would exists $(t_1, x_1) \in [0, 1] \times \partial\mathcal{A}(\beta, b, \alpha, a) \cap \Omega$ such that $Tx_1 + H(t_1, x_1) = x_1$. Since $x_1 \in \partial\mathcal{A}(\beta, b, \alpha, a) = \overline{\mathcal{A}} \cap \partial\mathcal{P}(\beta, b, \alpha, a)$, so $x_1 \in \partial\mathcal{P}(\beta, b, \alpha, a)$ we have that $\beta(x_1) = b$ or $\alpha(x_1) = a$.

(1) $\beta(x_1) = b$. Either $\psi(Tx_1 + Fx_1) < d$ or $\psi(Tx_1 + Fx_1) \geq d$.

If $\psi(Tx_1 + Fx_1) < d$, by the condition $(\mathcal{H}_3)$ we have that $\beta(Tx_1 + Fx_1) < b$, thus from the convexity of β and the condition $(\mathcal{H}_5)$ it follows that

$$\begin{aligned} b = \beta(x_1) &= \beta(Tx_1 + H(t_1, x_1)) \\ &= \beta(Tx_1 + t_1 Fx_1 + (1 - t_1)\omega_0) \\ &= \beta(t_1(Tx_1 + Fx_1) + (1 - t_1)(Tx_1 + \omega_0)) \\ &\leq t_1 \beta(Tx_1 + Fx_1) + (1 - t_1)\beta(Tx_1 + \omega_0) \\ &< t_1 b + (1 - t_1) b \\ &= b, \end{aligned}$$

which is a contradiction.

If $\psi(Tx_1 + Fx_1) \geq d$, we have that $\psi(x_1) \geq d$ since

$$\begin{aligned} \psi(x_1) &= \psi(Tx_1 + H(t_1, x_1)) \\ &= \psi(Tx_1 + t_1 Fx_1 + (1 - t_1)\omega_0) \\ &= \psi(t_1(Tx_1 + Fx_1) + (1 - t_1)(Tx_1 + \omega_0)) \\ &\geq t_1 \psi(Tx_1 + Fx_1) + (1 - t_1)\psi(Tx_1 + \omega_0) \\ &\geq d, \end{aligned}$$

and thus by the condition $(\mathcal{H}_3)$ we have that $\beta(Tx_1 + Fx_1) < b$, which is the same contradiction we arrived at in the previous subcase.

(2) $\alpha(x_1) = a$. Either $\theta(Tx_1 + Fx_1) \leq c$ or $\theta(Tx_1 + Fx_1) > c$.

If $\theta(Tx_1 + Fx_1) > c$, by the condition $(\mathcal{H}_2)$ we have that $\alpha(Tx_1 + Fx_1) > a$, thus from the concavity of α and the condition $(\mathcal{H}_4)$, it follows that

$$\begin{aligned} a = \alpha(x_1) &= \alpha(Tx_1 + H(t_1, x_1)) \\ &= \alpha(Tx_1 + t_1 Fx_1 + (1 - t_1)\omega_0) \\ &= \alpha(t_1(Tx_1 + Fx_1) + (1 - t_1)(Tx_1 + \omega_0)) \end{aligned}$$

$$\begin{aligned} &\geq t_1\alpha(Tx_1 + Fx_1) + (1-t_1)\alpha(Tx_1 + \omega_0) \\ &> a, \end{aligned}$$

which is a contradiction.

If $\theta(Tx_1 + Fx_1) \leq c$, we have that $\theta(x_1) \leq c$ since

$$\begin{aligned} \theta(x_1) &= \theta(Tx_1 + H(t_1, x_1)) \\ &= \theta(Tx_1 + t_1 Fx_1 + (1-t_1)\omega_0) \\ &= \theta(t_1(Tx_1 + Fx_1) + (1-t_1)(Tx_1 + \omega_0)) \\ &\leq t_1\theta(Tx_1 + Fx_1) + (1-t_1)\theta(Tx_1 + \omega_0) \\ &\leq c, \end{aligned}$$

and thus by the condition $(\mathcal{H}_2)$ we have that $\alpha(Tx_1 + Fx_1) > a$, which is the same contradiction we arrived at in the previous subcase.

Therefore, $Tx + H(t,x) \neq x$, for all $t \in [0,1]$ and $x \in \partial\mathcal{A}(\beta, b, \alpha, a) \cap \Omega$.

By the homotopy invariance property and the normality property of the fixed point index i_*

$$i_*(T + F, \mathcal{A}(\beta, b, \alpha, a), \mathcal{P}) = i_*(T + \omega_0, \mathcal{A}(\beta, b, \alpha, a), \mathcal{P}) = 1.$$

Then $T + F$ has at least one fixed point in $\mathcal{A}(\beta, b, \alpha, a)$. $\square$

6.3 Application to ODE's

In this section, we will investigate the equation

$$x''(t) + f(t, x(t)) = 0, \quad t \in (0,1), \tag{6.3}$$

with boundary conditions

$$x(0) = x'(1) = 0, \tag{6.4}$$

where $f : (0,1) \times [0,\infty) \to [0,\infty)$ is a continuous function.

The Green's function G for $-x''(t) = 0,\ t \in (0,1)$ satisfying (6.4) is given by

$$G(t,s) \ = \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1, \end{cases}$$

$$= \min\{t,s\},\ (t,s) \in [0,1] \times [0,1].$$

Note that, if x is a fixed point of the operator S defined by

$$Sx(t) = \int_0^1 G(t,s) f(s,x(s))\, ds,$$

then x is a solution of the boundary value problem (6.3)-(6.4).

6.3.1 Assumptions

Suppose that

($\mathbf{H}_1$) The functions f satisfy $\tilde{A} \le f(t,x(t)) \le a_1(t) + a_2(t)|x(t)|^p$, for $t \in [0,1]$ and $a_1, a_2 \in \mathcal{C}([0,1])$, $0 < a_1(t), a_2(t) \le A$ for $t \in [0,1]$, where p, A and $\tilde{A}$ are nonnegative constants.

($\mathbf{H}_2$) There exist positive constants $\varepsilon, \eta, a, b, c, d, C_1, C_2, C_3, \rho$ and R such that

$$\begin{gathered} \varepsilon \in (0,1), \quad \eta \in (0,1), \quad 2b \le \min(R,\rho), \\ A(1+b^p) < b, \\ \max\left(\frac{a}{2}, \frac{d}{C_2}\right) < C_1 < \min\left(b, \frac{c}{C_3}\right), \\ 0 \le C_3\left((1-\varepsilon)\frac{1}{\Lambda}(a - C_1) + \varepsilon C_1\right) \le c, \quad C_2\left((1-\varepsilon)\Lambda b + \varepsilon C_1\right) \ge d, \end{gathered} \tag{6.5}$$

where $\Lambda = \frac{\min(\frac{\eta}{8}\tilde{A}, C_1)}{\rho}$.

Remark 6.1. *As in [72, Remark 4.1], we can discuss the validity of the inequality (6.5).*

6.3.2 Main result

Our main result in this section is as follows.

Theorem 6.2. *If* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold, then the problem* (6.3)-(6.4) *has at least one positive solution* $x \in \mathcal{C}([0,1])$ *such that* $a < \min\limits_{t\in[0,1]} x(t) + 2C_1$ *and* $\max\limits_{t\in[0,1]} x(t) < b$.

To prove this result we will use theorem 6.1.

Proof. Define the Banach space $E = \mathcal{C}([0,1])$ with the norm $\|x\| = \max\limits_{t\in[0,1]} |x(t)|$ and

$$\mathcal{P} = \{x \in E : \; x(t) \geq 0, \, t \in [0,1], \; \min_{t\in[\frac{\eta}{8},\eta]} x(t) \geq \Lambda \|x\|\}, \quad \Omega = \{x \in \mathcal{P} : \; \|x\| \leq \rho\}.$$

For $x \in \mathcal{P}$ define the convex functionals

$$\beta(x) = \max_{t\in[0,1]} |x(t)|, \qquad \theta(x) = C_3 \max_{t\in[0,1]} |x(t)|.$$

For $x \in \mathcal{P}$ define the concave functionals

$$\alpha(x) = \min_{t\in[0,1]} x(t) + 2C_1, \qquad \psi(x) = C_2 \min_{t\in[\frac{\eta}{8},\eta]} x(t).$$

For $x \in \mathcal{P}$ define the operators

$$Tx(t) = (1-\varepsilon)x(t) + (\varepsilon - 1)C_1$$

$$Fx(t) = \varepsilon \int_0^1 G(t,s) f(s, x(s))\, ds + (1-\varepsilon)C_1, \quad t \in [0,1].$$

Note that if $x \in \mathcal{P}$ is a fixed point of the operator $T + F$ then it is a positive solution to the problem (6.3)-(6.4).

We set

$$\mathcal{A} = \mathcal{P} \cap B(0,R) = \{x \in \mathcal{P} : \; \|x\| < R\},$$

$$\mathcal{A}(\beta, b, \alpha, a) = \{x \in \mathcal{A} : \; a < \alpha(x) \text{ and } \beta(x) < b\}.$$

1. For $x, y \in \mathcal{P}$, we have

$$|(I-T)x(t) - (I-T)y(t)| = \varepsilon |x(t) - y(t)|, \; t \in [0,1].$$

Hence

$$\|(I-T)x-(I-T)y\| = \varepsilon\|x-y\|.$$

Thus, $I-T:\mathcal{P}\to E$ is Lipschitz invertible with a constant $\gamma=\frac{1}{\varepsilon}$.

2. Since f is continuous, then F is continuous. Also, for $x\in\mathcal{P}$, we have

$$\begin{aligned}|Fx(t)| &= |\varepsilon\int_0^1 G(t,s)f(s,x(s))\,ds + C_1 - \varepsilon C_1|\\ &\le \varepsilon(a_1(t)+a_2(t)|x(t)|^p)+C_1\\ &\le \varepsilon\, A(1+|x(t)|^p)+C_1\\ &\le \varepsilon\, A(1+\|x\|^p)+C_1<\infty, \quad t\in[0,1]\end{aligned}$$

and

$$\begin{aligned}|(Fx)'(t)| &= |\varepsilon\int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,x(s))\,ds + C_1 - \varepsilon C_1|\\ &\le |\varepsilon(a_1(t)+a_2(t)|x(t)|^p)+C_1|\\ &\le \varepsilon\, A(1+|x(t)|^p)+C_1\\ &\le \varepsilon\, A(1+\|x\|^p)+C_1<\infty, \quad t\in[0,1].\end{aligned}$$

Consequently, by Ascoli-Arzelà compactness criteria, $F:\mathcal{P}\to E$ is a completely continuous operator. Then $F:\mathcal{P}\to E$ is a 0-set contraction.

3. Let $\omega_0\equiv C_1\in\mathcal{A}(\beta,b,\alpha,a)\cap\mathcal{A}(\theta,c,\psi,d)$ be a constant, we have

$$(I-T)x=\varepsilon x+(1-\varepsilon)C_1$$

$$(I-T)^{-1}x=\frac{x-C_1}{\varepsilon}+C_1,\ x\in\mathcal{P}.$$

Hence,

$$\begin{aligned}\left|(I-T)^{-1}\omega_0(t)\right| = \left|(I-T)^{-1}C_1\right| &= \left|\frac{C_1-C_1}{\varepsilon}+C_1\right|\\ &= C_1\\ &< R, \quad t\in[0,1],\end{aligned}$$

and

$$\min_{t\in[\frac{\eta}{8},\eta]}((I-T)^{-1}\omega_0(t))=\frac{C_1-C_1}{\varepsilon}+C_1=C_1\ge\Lambda C_1.$$

Also,

$$\begin{aligned}\alpha((I-T)^{-1}\omega_0) = \min_{t\in[0,1]}((I-T)^{-1}\omega_0(t)) + 2C_1 &= \min_{t\in[0,1]}((I-T)^{-1}C_1) + 2C_1 \\ &> \frac{C_1 - C_1}{\varepsilon} + 3C_1 \\ &> a,\end{aligned}$$

and

$$\begin{aligned}\beta((I-T)^{-1}\omega_0) = \max_{t\in[0,1]}|(I-T)^{-1}\omega_0(t)| &= \max_{t\in[0,1]}\left|\frac{C_1 - C_1}{\varepsilon} + C_1\right| \\ &= C_1 \\ &< b.\end{aligned}$$

Then

$$(I-T)^{-1}\omega_0 \in \mathcal{A}(\beta, b, \alpha, a).$$

4. We have $\mathcal{A}(\beta, b, \alpha, a) \subset \mathcal{P} \cap B(0, b)$, then $\mathcal{A}(\beta, b, \alpha, a)$ is bounded and by construction of $\mathcal{A}$, $\partial\mathcal{A} \cap \overline{\mathcal{A}(\beta, b, \alpha, a)} = \emptyset$. So, the condition $(\mathcal{H}_1)$ holds.

5. Let $x \in \mathcal{P}$ with $\alpha(x) = a$ and either $\theta(x) \leq c$ or $\theta(Tx + Fx) > c$. Then

$$\begin{aligned}\alpha(Tx + Fx) &= \min_{t\in[0,1]}(Tx(t) + Fx(t)) + 2C_1 \\ &> a.\end{aligned}$$

Consequently, the condition $(\mathcal{H}_2)$ holds.

6. Let $x \in \mathcal{P}$ with $\beta(x) = b$ and either $\psi(Tx + Fx) < d$ or $\psi(x) \geq d$. Then

$$\begin{aligned}\beta(Tx + Fx) &= \max_{t\in[0,1]}|Tx(t) + Fx(t)| \\ &= \max_{t\in[0,1]}\left|(1-\varepsilon)x(t) + \varepsilon\int_0^1 G(t,s)f(s,x(s))\,ds\right| \\ &\leq (1-\varepsilon)\max_{t\in[0,1]}|x(t)| + \varepsilon\max_{t\in[0,1]}\int_0^1 |G(t,s)||f(s,x(s))|\,ds \\ &\leq (1-\varepsilon)b + \varepsilon A(1 + \|x\|^p) \\ &\leq (1-\varepsilon)b + \varepsilon A(1 + b^p) \\ &< b.\end{aligned}$$

Consequently, the condition $(\mathcal{H}_3)$ holds.

7. Let $x \in \mathcal{P}$ with $\alpha(x) = a$. Then

$$\begin{aligned} \alpha(Tx + \omega_0) &= \min_{t\in[0,1]}(Tx(t) + C_1) + 2C_1 \\ &= \min_{t\in[0,1]}((1-\varepsilon)x(t) + (\varepsilon - 1)C_1 + C_1) + 2C_1 \\ &\geq (1-\varepsilon)\min_{t\in[0,1]} x(t) + \varepsilon C_1 + 2C_1 \\ &= (1-\varepsilon)(a - 2C_1) + (\varepsilon + 2)C_1 \\ &> a, \end{aligned}$$

and

$$\begin{aligned} \theta(Tx + \omega_0) &= C_3 \max_{t\in[0,1]} |Tx(t) + C_1| \\ &= C_3 \max_{t\in[0,1]} |(1-\varepsilon)x(t) + \varepsilon C_1| \\ &\leq C_3\left((1-\varepsilon)\max_{t\in[0,1]}|x(t)| + \varepsilon C_1\right) \\ &\leq C_3\left((1-\varepsilon)\frac{1}{\Lambda}\min_{t\in[\frac{\eta}{8},\eta]} x(t) + \varepsilon C_1\right) \\ &\leq C_3\left((1-\varepsilon)\frac{1}{\Lambda}(a - C_1) + \varepsilon C_1\right) \\ &\leq c. \end{aligned}$$

Consequently, the condition($\mathcal{H}_4$) holds.

8. Let $x \in \mathcal{P}$ with $\beta(x) = b$. Then

$$\begin{aligned} \beta(Tx + \omega_0) &= \max_{t\in[0,1]} |Tx(t) + C_1| \\ &= \max_{t\in[0,1]} |(1-\varepsilon)x(t) + \varepsilon C_1| \\ &\leq (1-\varepsilon)\max_{t\in[0,1]} |x(t)| + \varepsilon C_1 \\ &\leq (1-\varepsilon)b + \varepsilon C_1 \\ &< b, \end{aligned}$$

and

$$\begin{aligned}
\psi(Tx+\omega_0) &= C_2 \min_{t\in[\frac{\eta}{8},\eta]} (Tx(t)+C_1) \\
&= C_2 \left(\min_{t\in[\frac{\eta}{8},\eta]} (1-\varepsilon)x(t) + \varepsilon C_1 \right) \\
&\geq C_2 \left((1-\varepsilon) \min_{t\in[\frac{\eta}{8},\eta]} x(t) + \varepsilon C_1 \right) \\
&\geq C_2 \left((1-\varepsilon)\Lambda\|x\| + \varepsilon C_1 \right) \\
&\geq C_2 \left((1-\varepsilon)\Lambda b + \varepsilon C_1 \right) \\
&\geq d.
\end{aligned}$$

Consequently, the condition $(\mathcal{H}_5)$ holds.

9. Let $\lambda \in [0,1]$ is fixed and $\tilde{x} \in \mathcal{A}(\beta, b, \alpha, a)$ is arbitrary chosen. Set

$$w(t) = \lambda \int_0^1 G(t,s) f(s, \tilde{x}(s))\, ds + (1-\lambda) C_1.$$

We have that $w(t) \geq 0$, $t \in [0,1]$, and

$$w(t) \leq A(1+|\tilde{x}(t)|^p) + (1-\lambda)C_1 < A\left(1+b^p\right) + b, \quad t \in [0,1].$$

So,

$$\|w\| < A\left(1+b^p\right) + b \leq \rho,$$

and

$$\begin{aligned}
\min_{t\in[\frac{\eta}{8},\eta]} w(t) &= \min_{t\in[\frac{\eta}{8},\eta]} \left(\lambda \int_0^1 G(t,s) f(s,\tilde{x}(s))\, ds + (1-\lambda)C_1 \right) \\
&\geq \lambda \frac{\eta}{8} \tilde{A} + (1-\lambda) C_1 \\
&\geq \frac{\min(\frac{\eta}{8}\tilde{A}, C_1)}{\rho} \rho \\
&\geq \Lambda \|w\|.
\end{aligned}$$

Therefore $\omega \in \Omega$. Also, we have

$$\begin{aligned}
\lambda F\tilde{x}(t) + (1-\lambda)\omega_0 &= \lambda\left[\varepsilon\int_0^1 G(t,s)f(s,\tilde{x}(s))\,ds + (1-\varepsilon)C_1\right] + (1-\lambda)C_1 \\
&= \lambda\varepsilon\int_0^1 G(t,s)f(s,\tilde{x}(s))\,ds + \lambda(1-\varepsilon)C_1 + (1-\lambda)C_1 \\
&= \lambda\varepsilon\int_0^1 G(t,s)f(s,\tilde{x}(s))\,ds - \lambda\varepsilon C_1 + C_1 \\
&= \lambda\varepsilon\int_0^1 G(t,s)f(s,\tilde{x}(s))\,ds - \lambda\varepsilon C_1 + C_1 - \varepsilon C_1 + \varepsilon C_1 \\
&= \varepsilon\left(\lambda\int_0^1 G(t,s)f(s,\tilde{x}(s))\,ds + (1-\lambda)C_1\right) + (1-\varepsilon)C_1 \\
&= \varepsilon w(t) + (1-\varepsilon)C_1 \\
&= (I-T)w(t), \quad t\in[0,1].
\end{aligned}$$

Then

$$\lambda F(\mathcal{A}(\beta,b,\alpha,a)) + (1-\lambda)\omega_0 \subset (I-T)(\Omega), \text{ for all } \lambda\in[0,1].$$

Hence, all the conditions of Theorem 6.1 are satisfied, and it follows that the operator $T+F$ has at least one fixed point $x^* \in \mathcal{A}(\beta,b,\alpha,a)$, which is a positive solution of the problem (6.3)-(6.4). This completes the proof. □

6.3.3 Example

Let,

$$\varepsilon = \eta = \frac{1}{2}, \quad a = 8, \quad b = 10, \quad c = 12, \quad d = \frac{1}{3}, \quad C_1 = 8, \quad C_2 = C_3 = 1,$$

$$A = 2, \quad \tilde{A} = \frac{1}{4},$$

$$p = \frac{1}{2}, \quad R = 25, \quad \rho = 25.$$

So,

$$2b = 20 \le \min(\rho, R) = 25, \quad A(1+b^p) = 2(1+10^{\frac{1}{2}}) = 8.3246 < b = 10,$$

$$\max\left(\frac{a}{2}, \frac{d}{C_2}\right) = \max\left(4, \frac{1}{3}\right) = 4 < C_1 = 8 < \min\left(b, \frac{c}{C_3}\right) = \min(10,12) = 10,$$

$$\Lambda = \frac{\min(\frac{\eta}{8}\tilde{A}, C_1)}{\rho} = 6.25\times 10^{-4},$$

$$C_3\left((1-\varepsilon)\frac{1}{\Lambda}(a-C_1) + \varepsilon C_1\right) = 4 \le c = 12,$$

$$C_2\left((1-\varepsilon)\Lambda b+\varepsilon C_1\right)=4.0031\geq d=\frac{1}{3}.$$

Thus, $(\mathbf{H}_2)$ holds. Now, by our main result, it follows that the boundary value problem:

$$\begin{cases} x''(t)=\frac{t+1}{4}+2\frac{\sqrt{x(t)}}{1+x(t)^4}, & t\in(0,1),\\ x(0)=x'(1)=0, \end{cases}$$

has at least one positive solution.

6.4 Concluding remarks

(1) In this chapter, we have obtained a new functional fixed point theorem on intervals of functional type, using the fixed point index approach for the sum of two operators on cones of a Banach space.

(2) We discussed existence of at least one positive solution to the problem (6.3)-(6.4) using the developed fixed point theorem for the sum of two operators, The solutions are localized in functional type intervals of a cone $\mathcal{A}(\beta,b,\alpha,a)$ instead of sets of the form $\mathcal{P}(\beta,b,\alpha,a)$.

7 Fourth order singular eigenvalue boundary value problems

The results of this chapter are obtained by Bouchal, Mebarki and Georgiev in [23].

7.1 Introduction

In this chapter, we investigate the following fourth order singular differential equation with parameter with boundary conditions at two points

$$v^{(4)}(t) = \lambda g(t) f(v(t)), \quad 0 < t < 1, \tag{7.1}$$

$$v(0) = a_1, \quad v(1) = a_2, \quad v''(0) = a_3, \quad v''(1) = a_4, \tag{7.2}$$

where $a_i \geq 0$, $i \in \{1, 2, 3, 4\}$, are given constants. Assume that we have the two following hypotheses:

$(\mathcal{H}_1)$ $f \in \mathcal{C}([0, \infty))$,

$$0 < A_1 \leq f(x) \leq A_2 + \sum_{j=0}^{k} B_j x^j, \quad x \in [0, \infty),$$

$A_2 \geq A_1 > 0$ and $B_j \geq 0$, $j \in \{0, \ldots, k\}$, $k \in \mathbb{N}_0$, are given constants.

$(\mathcal{H}_2)$ $g : (0, 1) \to \mathbb{R}^+$ is continuous and may be singular at $t = 0$ or/and $t = 1$, $g \not\equiv 0$ on $(0, 1)$ and $\int\limits_0^1 s(1-s)g(s)ds < \infty$.

Fourth order two point boundary value problems (BVPs for short) have been received much attention by many authors due to their importance in physics. Usually, they are essential in describing a vast class of elastic deflections with several types of boundary conditions such as whose ends are simply-supported at 0 and 1 ($v(0) = v(1) = v''(0) = v''(1) = 0$). A great number of research has been devoted to investigate the existence of positive solutions to this class of problems, see [15, 14, 17, 49, 65, 68, 77, 90, 95] and the references therein. The authors in [15] discussed the existence, uniqueness and multiplicity of positive solutions to the following eigenvalue BVP by means of fixed point theorem and degree theory

$$v^{(4)}(t) = \lambda f(t, (v(t)), \quad 0 < t < 1, \tag{7.3}$$

$$v(0) = v(1) = v''(0) = v''(1) = 0, \tag{7.4}$$

where $\lambda > 0$ is a constant and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. In [90] by applying a Krasnosel'skii fixed point theorem of cone expansion and compression the author obtained the existence and multiplicity results of equation (7.3) with boundary conditions $v(0) = v(1) = v'(0) = v'(1) = 0$. In the literature, there are few papers devoted to study fourth order singular eigenvalue problems. In the case when $a_i = 0$, $i \in \{1, 2, 3, 4\}$, the BVP (7.1)-(7.2) is investigated in [36] when $f \in \mathcal{C}([0, \infty))$, $f > 0$ on $[0, \infty)$, f is nondecreasing on $[0, \infty)$ and there exist $\delta > 0$, $m \geq 2$ such that $f(u) > \delta u^m$, $u \in [0, \infty)$, and $g \in \mathcal{C}(0, 1)$, $g > 0$ on $(0, 1)$ and $0 < \int\limits_0^1 s(1 - s)g(s)ds < \infty$. In [36], Feng and Ge used the method of upper and lower solutions and the fixed point index to discuss the existence of positive solutions.

The approach used in this chapter is to rewrite the BVP (7.1)-(7.2) into a perturbed integral equation for which we search for solutions in a suitable subset of a Banach space by means of recent fixed point theorem of Birkhoff-Kellogg type developed by Calamai and Infante in [27]. Note that this fixed point theorem has been applied very recently to discuss the solvability of fourth order retarded equations in [26].

7.2 Birkhoff-Kellogg type fixed point theorem in cones

The possible existence of a positive eigenvalue plays an important role in the study of nonlinear operators. One of the classical results concerning the existence of positive eigenvalues for compact mappings originates from the work of Birkhoff and Kellogg [20], and it appears in the publication of Schauder ([79], page 180) as follows:

Theorem 7.1. ***(The Birkhoff-Kellogg Theorem)*** *Let B be the closed unit ball of an infinite dimensional Banach space E and $F : \partial B \to E$ a compact continuous single valued function such that $F(\partial B)$ has a positive distance from 0. Then F has an invariant direction, i.e. there is $x \in \partial B$ and $\lambda > 0$ such that $F(x) = \lambda x$.*

Krasnosel'skii and Ladyzenskii introduced the Birkhoff-Kellogg type theorem in cones in 1954 [60], and recently on 2022 [27] Calmai and Infante give another version of this theorem on translate of cone, which is a kind of complement result of the interesting topological results proved by Djebali and Mebarki in [33].

Let $(X, \|.\|)$ be a real Banach space. For a given $y \in X$, we consider the translate of a cone $\mathcal{K}$, namely

$$\mathcal{K}_y = \mathcal{K} + y = \{x + y \ : x \in \mathcal{K}\}.$$

Given an open bounded subset D of X we denote $D_{\mathcal{K}_y} = D \cap \mathcal{K}_y$, an open subset of $\mathcal{K}_y$. The following theorem will be used in Chapter 7 to prove the existence of at least one positive solution to a fourth order boundary value problem with parameter.

Theorem 7.2. *[27, Corollary 2.4] Let $(X, \|.\|)$ be a real Banach space, $\mathcal{K} \subset X$ be a cone, and $D \subset X$ be an open bounded set with $y \in D_{\mathcal{K}_y}$ and $\overline{D}_{\mathcal{K}_y} \neq \mathcal{K}_y$. Assume that $F : \overline{D}_{\mathcal{K}_y} \to \mathcal{K}$ is a completely continuous map and assume that*

$$\inf_{x \in \partial D_{\mathcal{K}_y}} \|Fx\| > 0.$$

Then there exist $x^* \in \partial D_{\mathcal{K}_y}$ and $\lambda^* \in (0, \infty)$ such that

$$x^* = y + \lambda^* F(x^*).$$

7.3 Auxiliary results

Let

$$y_1(t) = (a_1 + \frac{a_4}{6})(1 - t) + a_2 t + \frac{a_3}{6}(1 - t)^3 + \frac{a_4}{6}(t^3 - 1) + \frac{a_3}{6}(t - 1), \quad t \in [0, 1].$$

We have

$$0 \le y_1(t) \le a_1 + a_2 + a_3 + a_4, \quad t \in [0,1],$$

and

$$\begin{aligned} y_1'(t) &= -a_1 - \frac{a_4}{6} + a_2 - \frac{1}{2}a_3(1-t)^2 + \frac{1}{2}a_4 t^2 + \frac{a_3}{6}, \quad t \in [0,1]. \\ y_1''(t) &= a_3(1-t) + a_4 t, \quad t \in [0,1]. \end{aligned}$$

Hence,

$$y_1(0) = a_1, \quad y_1(1) = a_2, \quad y_1''(0) = a_3, \quad y_1''(1) = a_4.$$

Set

$$y(t) = -y_1(t), \quad t \in [0,1].$$

Now, consider the BVP

$$\begin{aligned} u^{(4)} &= \lambda g(t) f(u(t) - y(t)), \quad 0 < t < 1, \\ u(0) &= u(1) = u''(0) = u''(1) = 0, \end{aligned} \tag{7.5}$$

where f and g satisfy $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, respectively.

Let $X = \mathcal{C}([0,1])$ be endowed with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define the cone

$$\mathcal{K} = \{u \in X : u(t) \ge 0, \quad t \in [0,1]\}.$$

Since $0 \le \int_0^1 s(1-s)g(s)ds < \infty$, there exists a nonnegative constant C_0 such that

$$\int_0^1 s(1-s)g(s)ds = C_0.$$

Because $g \not\equiv 0$ on $(0,1)$, there are $C_1 > 0$, $s_0 \in (0,1)$ and $\epsilon > 0$ such that $s_0 - \epsilon, s_0 + \epsilon \in (0,1)$ and

$$g(s) \ge C_1, \;\; s \in (s_0 - \epsilon, s_0 + \epsilon).$$

Define

$$G(t,s) = \begin{cases} t(1-s)\frac{2s - s^2 - t^2}{6}, & 0 \le t \le s \le 1, \\ s(1-t)\frac{2t - t^2 - s^2}{6}, & 0 \le s \le t \le 1. \end{cases}$$

We have

$$0 \le G(t,s) \le \frac{1}{6}s(1-s) \le \frac{1}{6}, \quad 0 \le t, s \le 1,$$

Note that

$$\begin{aligned}
\int_0^1 G(s_0+\epsilon,s)g(s)ds &\geq \int_{s_0-\epsilon}^{s_0+\epsilon} G(s_0+\epsilon,s)g(s)ds\\
&\geq C_1\int_{s_0-\epsilon}^{s_0+\epsilon} G(s_0+\epsilon,s)ds\\
&= C_1\int_{s_0-\epsilon}^{s_0+\epsilon} s(1-s_0-\epsilon)\frac{2(s_0+\epsilon)-(s_0+\epsilon)^2-s^2}{6}ds\\
&\geq \frac{2}{3}C_1\epsilon(s_0-\epsilon)^2(1-s_0-\epsilon)^2\\
&> 0.
\end{aligned}$$

For $u\in X$, define the operator

$$Tu(t)=\int_0^1 G(t,s)g(s)f(u(s)-y(s))ds,\quad t\in[0,1].$$

In [36], it is proved that any fixed point $u\in X$ of the operator λT is a solution to the BVP (7.5). Fix $C_2>a_1+a_2+a_3+a_4$ arbitrarily. Define

$$D=\{u\in X:\|u\|<C_2\}.$$

We have that D is an open bounded set in X, $y\in D$ and $D_{\mathcal{K}_y}=D\cap\mathcal{K}_y\neq\mathcal{K}_y$, with $\mathcal{K}_y=\mathcal{K}+y=\{x+y\ :x\in\mathcal{K}\}$.. Note that for any $u\in\overline{D}_{\mathcal{K}_y}$, we have $u(t)=y(t)+z(t)$, $t\in[0,1]$, for some $z\in\mathcal{K}$, and so $u(t)-y(t)=z(t)\geq 0$, $t\in[0,1]$, and

$$\begin{aligned}
f(u(t)-y(t)) &\leq \left(A_2+\sum_{j=0}^k B_j(u(t)-y(t))^j\right)\\
&\leq \left(A_2+\sum_{j=0}^k B_j2^j\left(|u(t)|^j+|y_1(t)|^j\right)\right)\\
&\leq \left(A_2+\sum_{j=0}^k B_j2^j\left(C_2^j+(a_1+a_2+a_3+a_4)^j\right)\right),\quad t\in[0,1].
\end{aligned}$$

7.4 Main result

Our main result is as follows

Theorem 7.3. *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold. Then there is a* $\lambda^*>0$ *such that the BVP* (7.1)-(7.2) *has at least one positive solution for* $\lambda=\lambda^*$.

To prove this result we will use the Theorem 7.2.

Proof. Since $f \in \mathcal{C}([0,\infty))$ and $g \in \mathcal{C}(0,1)$, we have that $T : D_{\mathcal{K}_y} \to \mathcal{K}$ is a continuous operator. Next, for $u \in \overline{D}_{\mathcal{K}_y}$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)g(s)f(u(s)-y(s))ds \\ &\le \frac{1}{6}\left(A_2 + \sum_{j=0}^{k} B_j 2^j \left(C_2^j + (a_1+a_2+a_3+a_4)^j\right)\right)\int_0^1 s(1-s)g(s)\,ds \\ &= \frac{1}{6}C_0\left(A_2 + \sum_{j=0}^{k} B_j 2^j \left(C_2^j + (a_1+a_2+a_3+a_4)^j\right)\right), \quad t\in[0,1], \end{aligned}$$

whereupon

$$\|Tu\| \le C_0\left(A_2 + \sum_{j=0}^{k} B_j 2^j \left(C_2^j + (a_1+a_2+a_3+a_4)^j\right)\right).$$

Then, $T(\overline{D}_{\mathcal{K}_y})$ is uniformly bounded. Moreover, for $u \in \overline{D}_{\mathcal{K}_y}$ and $t_1, t_2 \in [0,1], t_1 < t_2$, the Lebesgue dominated convergence theorem guarantees that

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| &\le \int_0^1 |G(t_1,s)-G(t_2,s)|g(s)f(u(s)-y(s))ds\,ds \\ &\le \left(A_2 + \sum_{j=0}^{k} B_j 2^j \left(C_2^j + (a_1+a_2+a_3+a_4)^j\right)\right)\int_0^1 g(s)|G(t_1,s)-G(t_2,s)|ds \\ &\to 0, \quad t_1 \to t_2, \end{aligned}$$

Therefore, $T(\overline{D}_{\mathcal{K}_y})$ is equicontinuous. According to the Ascoli-Arzelà compactness criterion, we conclude that the operator $T : \overline{D}_{\mathcal{K}_y} \to \mathcal{K}$ is completely continuous.

Observe that, for $u \in \partial D_{\mathcal{K}_y}$,

$$\begin{aligned} \max_{t\in[0,1]} |Tu(t)| \ge Tu(s_0+\epsilon) &= \int_0^1 G(s_0+\epsilon,s)g(s)f(u(s)-y(s))ds \\ &\ge A_1\int_0^1 G(s_0+\epsilon,s)g(s)ds \\ &\ge \frac{2}{3}A_1C_1\epsilon(s_0-\epsilon)^2(1-s_0-\epsilon)^2 \\ &> 0. \end{aligned}$$

Consequently

$$\inf_{u\in\partial D_{\mathcal{K}_y}} \|Tu\| \ge \frac{2}{3}A_1C_1\epsilon(s_0-\epsilon)^2(1-s_0-\epsilon)^2 > 0.$$

Now, applying Theorem 7.2, we conclude that there are $\lambda^* \in (0,\infty)$ and $u^* \in \partial D_{\mathcal{K}_y}$ such that

$$u^*(t) = y(t) + \lambda^*\int_0^1 G(t,s)g(s)f(u^*(s)-y(s))ds, \quad t\in[0,1].$$

Let

$$v^*(t) = u^*(t) - y(t), \quad t \in [0,1].$$

Then

$$\begin{aligned} v^*(0) &= u^*(0) - y(0) = a_1, \\ v^*(1) &= u^*(1) - y(1) = a_2, \\ v^{*\prime\prime}(0) &= u^{*\prime\prime}(0) - y''(0) = a_3, \\ v^{*\prime\prime}(1) &= u^{*\prime\prime}(1) - y''(1) = a_4 \end{aligned}$$

and

$$v^*(t) = \lambda \int_0^1 G(t,s)g(s)f(v^*(s))ds, \quad t \in [0,1],$$

whereupon

$$v^{*(4)}(t) = \lambda g(t) f(v^*(t)), \quad 0 < t < 1.$$

Since $u^* \in \partial D_{\mathcal{K}_y}$, we have that $u^*(t) = y(t) + z^*(t)$, $t \in [0,1]$, for some $z^* \in \mathcal{K}$, and then

$$v^*(t) = u^*(t) - y(t) = z^*(t) + y(t) - y(t) = z^*(t) \geq 0, \quad t \in [0,1].$$

$\square$

7.5 Example

Consider the following BVP

$$\begin{aligned} u^{(4)} &= \lambda \frac{\left(\frac{1}{2}-t\right)^2}{t(1-t)} \left(1 + \frac{1}{1+(u(t))^2}\right), \quad t \in (0,1), \\ u(0) &= 0, \quad u(1) = 1, \quad u''(0) = \tfrac{1}{2}, \quad u''(1) = 1. \end{aligned} \tag{7.6}$$

Here

$$f(x) = 1 + \frac{1}{1+x^2}, \quad x \in [0,\infty), \quad g(t) = \frac{\left(\frac{1}{2}-t\right)^2}{t\,(1-t)}, \quad t \in (0,1),$$

and

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = 1.$$

By our main result, it follows that the BVP (7.6) has at least one positive solution.

7.6 Concluding remarks

(1) In our main result we do not require any monotonicity assumptions on f, and we do not assume that f is either superlinear or sublinear.

(2) In the particular case $a_i = 0$, $i \in \{1, 2, 3, 4\}$, our main result is valid in the case when f is decreasing on $[0, \infty)$, while the corresponding result in [36] is not valid. For instance, $f(x) = 1 + \frac{1}{1+x^2}$, $x \in [0, \infty)$, satisfies $(\mathcal{H}_1)$ for $A_1 = 1$, $A_2 = 2$, $B_j = 0$, $j \in \{0, \ldots, k\}$, and f is decreasing on $[0, \infty)$, whereupon it does not satisfy the conditions in [36]. Also, the conditions for g in [36] are more restrictive than $(\mathcal{H}_2)$. For instance, $g(t) = \frac{\left(\frac{1}{2}-t\right)^2}{t(1-t)}$, $t \in (0, 1)$, satisfies $(\mathcal{H}_2)$ and does not satisfy the conditions in [36] because $g\left(\frac{1}{2}\right) = 0$. Thus, we can consider the particular case of our main result, $a_i = 0$, $i \in \{1, 2, 3, 4\}$, as a complementary result to the result in [36].

Annexes

A Sum and integral formulations

One of the methods used to study boundary value problems is the fixed point theory, where we transform the problem into a fixed point equation for a suitable operator. If L is a linear operator and the equation $Lu(t) = h(t), \forall t \in [a, b]$ is associated to appropriate homogeneous linear boundary value conditions, has only the trivial solution $u \equiv 0$ for $h \equiv 0$, then the operator L is invertible and its inverse operator L^{-1} is characterized by an integral kernel, $G(t, s)$ called Green's function and the solution of the considered problem is then given by

$$u(t) = L^{-1}f(t) = \int_a^b G(t, s)h(s)\, ds, \quad t \in [a, b].$$

George Green $(1793 - 1841)$ was the first who introduced such kernels to solve boundary value problems. The principle advantage of Green's function is the fact that it is independent of the function h called usually the second member. To get the exact solution for each case of h we only need to calculate the corresponding integral, and so we have the expression that we are looking for.

The sum and integral formulations of the boundary value problems studied in this thesis are presented in this section; the first subsection is devoted to difference boundary value problems, and the second to differential boundary value problems.

A.1 Sum formulation of problems (3.1) and (4.7)-(4.8)

I. Impulsive difference equations with linear two point boundary conditions

Consider the following boundary value problem for first impulsive difference equations at two point boundary conditions

$$\begin{aligned} \Delta u(n) + cu(n) &= \sigma(n), \quad n \neq n_k, \quad n \in J, \\ \Delta u(n_k) &= -L_k u(n_k) + I_k(\eta(n_k)) + L_k \eta(n_k), \quad k \in \{1, \ldots, p\}, \\ Mu(0) - Nu(T) &= C, \end{aligned} \tag{7}$$

where $0 < c < 1$, L_k, C, $k \in \{1, \ldots, p\}$, are given constants, $\eta \in E_1, \sigma \in \mathcal{C}(J)$, where E_1 is the set of real-valued functions defined on J, where Δ is the forward difference operator, i.e., $\Delta u(n) = u(n+1) - u(n)$, $J = [0, T] \cap \mathbb{N}$, $T \in \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers, $M, N > 0$, $I_k \in \mathcal{C}(\mathbb{R})$, $k \in \{1, \ldots, p\}$, $\{n_k\}_{k=1}^p$ are fixed impulsive points such that

$$0 < n_1 < n_2 < \ldots < n_p < T, \quad p \in \mathbb{N}.$$

The solution of the problem (7) can be represented in the form

$$\begin{aligned} u(n) &= \frac{C(1-c)^n}{M - N(1-c)^T} + \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) \\ &\quad + \sum_{0<n_k \leq T-1} G(n, n_k)\left((c - L_k)u(n_k) + I_k(\eta(u_k)) + L_k\eta(u_k)\right), \end{aligned} \tag{8}$$

where

$$G(n,j) = \frac{1}{M - N(1-c)^T} \begin{cases} M\frac{(1-c)^n}{(1-c)^{j+1}}, & 0 \leq j \leq n-1, \\ N\frac{(1-c)^{T+n}}{(1-c)^{j+1}}, & n \leq j \leq T-1. \end{cases}$$

Proof. Set $y(n) = \frac{u(n)}{(1-c)^n}$, $n \in J$. From (7), we see that $y(n)$ satisfies

$$\begin{aligned} y(n+1) &= y(n) + \tfrac{\sigma(n)}{(1-c)^{n+1}}, \quad n \neq n_k, \quad n \in J, \\ \Delta y(n_k) &= \tfrac{c - L_k}{1-c} y(n_k) + \tfrac{I_k(\eta(n_k)) + L_k\eta(n_k)}{(1-c)^n_k+1}, \quad k \in \{1, \ldots, p\}, \\ My(0) &- Ny(T)(1-c)^T = C, \end{aligned} \tag{9}$$

From (9), we have

$$y(n) = y(0) + \sum_{j=0, j\neq n_k}^{n-1} \frac{\sigma(j)}{(1-c)^{j+1}} + \sum_{0<n_k \leq n-1} \left(\frac{c - L_k}{1-c} y(n_k) + \frac{I_k(\eta(u_k)) + L_k\eta(u_k)}{(1-c)^{n_k+1}}\right). \tag{10}$$

Let $n=T$ in (10). Then we get

$$y(n)=y(0)+\sum_{j=0,j\neq n_k}^{T-1}\frac{\sigma(j)}{(1-c)^{j+1}}+\sum_{0<n_k\leq T-1}\left(\frac{c-L_k}{1-c}y(n_k)+\frac{I_k(\eta(u_k))+L_k\eta(u_k)}{(1-c)^{n_k+1}}\right). \qquad (11)$$

From the boundary conditions $y(T)=\frac{My(0)-C}{N(1-c)^T}$, we obtain

$$\begin{aligned} y(n)=\frac{C}{M-N(1-c)^T}+\frac{N(1-c)^T}{M-N(1-c)^T}\Bigg[\sum_{j=0,j\neq n_k}^{T-1}\frac{\sigma(j)}{(1-c)^{j+1}} \\ +\sum_{0<n_k\leq T-1}\left(\frac{c-L_k}{1-c}y(n_k)+\frac{I_k(\eta(u_k))+L_k\eta(u_k)}{(1-c)^{n_k+1}}\right)\Bigg] \end{aligned} \qquad (12)$$

Substituting (12) into (10) and using $y(n)=\frac{u(n)}{(1-c)^n}$, $n\in J$, we see that u satisfies (8). If u is a solution of (8) then u satisfies the boundary value problem (7). □

II. Second order difference equation

Consider the following second order difference equation with Dirichlet boundary conditions

$$\begin{aligned} \triangle^2 u(k)+f(k) &= 0, \quad k\in\{0,1,\ldots,N\},\ N\in\mathbb{N}, N>1. \\ u(0)=u(N+2) &= 0. \end{aligned} \qquad (13)$$

where $f:\{0,...,N+2\}\to[0,\infty)$ is a continuous function.

Δ^2 the second forward difference operator which acts on u by $\triangle^2 u(k)=u(k+2)-2u(k+1)+u(k)$, $k\in\{0,1,\ldots,N\}$.

From Theorem 2.16 the boundary value problem (13), is equivalent to the sum equation

$$u(k)=\sum_{l=1}^{N+1}H(k,l)f(l), \quad k\in\{0,\ldots,N+2\},$$

with

$$H(k,l)=\begin{cases} -\frac{y(N+2,l)}{y_1(N+2)}y_1(k), & k\leq l \\ y(k,l)-\frac{y(N+2,l)}{y_1(N+2)}y_1(k), & k\geq l \end{cases}$$

for any $l\in\{0,...,N+2\}$.

The Cauchy function

We have that $u_1(k)=1$, $u_2(k)=k$ are two linearly independent solution of the second order difference equation $\Delta^2 u(k)=0$. In fact,

$$W(u_1,u_2)=\begin{vmatrix} 1 & k \\ 1 & k+1 \end{vmatrix}=1\neq 0.$$

Then the Cauchy function is given by

$$y(k,l)=\frac{\begin{vmatrix} u_1(l) & u_2(l)\\ u_1(k) & u_2(k)\end{vmatrix}}{p(l)\begin{vmatrix} u_1(l) & u_2(l)\\ u_1(l+1) & u_2(l+1)\end{vmatrix}}=\frac{\begin{vmatrix} 1 & l\\ 1 & k\end{vmatrix}}{\begin{vmatrix} 1 & l\\ 1 & l+1\end{vmatrix}}=\frac{k-l}{l+1-l}=k-l.$$

The equation $\Delta^2 u(k) = 0$ can be written in the form (2.45) with $p(k) = 1$ and $q(k) = 0$. $y_1(k) = k$ is a solution to the following initial problem

$$\begin{cases} \Delta^2 y_1(k) = 0, \\ y_1(0) = 0, \\ y_1(1) = 1, \end{cases} \tag{14}$$

Consequently, From (2.51), we have

$$H(k,l)=\begin{cases} -\frac{y(N+2,l)}{y_1(N+2)}y_1(k) = -\frac{(N+2-l)}{N+2}k, & \text{for } k \leq l \\ y(k,l) - \frac{y(N+2,l)}{y_1(N+2)}y_1(k) = (k-l) - \frac{(N+2-l)}{N+2}k = -\frac{l(N+2-k)}{N+2}, & \text{for } k \geq l. \end{cases} \tag{15}$$

Hence, the Green's function associated to the operator $\Delta^2 u(k) = 0$ with Dirichlet boundary conditions is given by

$$H(k,l)=\frac{1}{N+2}\begin{cases} -k(N+2-l), & k \in \{0,\dots,l\}, \\ -l(N+2-k), & k \in \{l+1,\dots,N+2\}, \end{cases} \tag{16}$$

with $(k,l) \in [0, N+2] \times [1, N+1]$.

A.2 Integral formulation of problems (5.3) and (7.1)-(7.2)

I. Generalized Strum Liouville multipoint boundary value problem

Consider the following Sturm-Liouville multipoint boundary value problem

$$(\mathcal{P}_1)\begin{cases} (p(t)u'(t))' - q(t)u(t) &= -v(t), \quad 0 < t < 1, \\ au(0) - bu'(0) &= \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) &= \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{17}$$

where $a, b, c, d \in [0, \infty),\ 0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1\ (m \geq 3),\ a_i, b_i \in [0, \infty)$ are constants for $i = 1, 2, \ldots, m-2$ and $\rho = ac + ad + bc > 0$ and $p \in \mathcal{C}^1([0,1],(0,\infty)),\ q \in \mathcal{C}([0,1],(0,\infty))$.

Let $x(t) = at + b$ and $y(t) = d + c(1-t)$ for $t \in [0,1]$ be the solutions to the problems

$$\begin{cases} (p(t)x'(t))' - q(t)x(t) &= 0 \quad 0 < t < 1, \\ x(0) &= b, \\ x'(0) &= a, \end{cases}$$

$$\begin{cases} (p(t)y'(t))' - q(t)y(t) &= 0 \quad 0 < t < 1, \\ y(1) &= d, \\ y'(1) &= -c, \end{cases}$$

Set

$$\Delta := \begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i x(\xi_i) & \rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i) \\ \rho - \sum\limits_{i=1}^{m-2} b_i x(\xi_i) & -\sum\limits_{i=1}^{m-2} b_i y(\xi_i) \end{vmatrix}$$

and

$$\rho = p(0) \begin{vmatrix} y(0) & x(0) \\ y'(0) & x'(0) \end{vmatrix}.$$

Since x and y are linearly independent then by Liouville formula, we have that

$$\forall t \in [0,1] \quad \rho = p(t) \begin{vmatrix} y(t) & x(t) \\ y'(t) & x'(t) \end{vmatrix}.$$

If $\Delta \neq 0$, then the problem $(\mathcal{P}_1)$ has a unique solution given by the following integral equation

$$u(t) = \int_0^1 G(t,s)v(s)\,ds + \mathcal{A}(v)\,x(t) + \mathcal{B}(v)\,y(t), \quad t \in [0,1], \tag{18}$$

where

$$G(t,s) = \frac{1}{\rho} \begin{cases} y(t)x(s), & 0 \leq s \leq t \leq 1, \\ x(t)y(s), & 0 \leq t \leq s \leq 1, \end{cases}$$

$$= \frac{1}{\rho} \begin{cases} (d + c(1-t))(as + b), & 0 \leq s \leq t \leq 1, \\ (at + b)(d + c(1-s)), & 0 \leq t \leq s \leq 1, \end{cases}$$

with

$$\mathcal{A}(v) := \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)v(s)\,ds & \rho - \sum_{i=1}^{m-2} a_i y(\xi_i) \\ \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)v(s)\,ds & -\sum_{i=1}^{m-2} b_i y(\xi_i) \end{vmatrix}, \tag{19}$$

$$\mathcal{B}(v) := \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i x(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)v(s)\,ds \\ \\ \rho - \sum_{i=1}^{m-2} b_i x(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)v(s)\,ds \end{vmatrix}. \tag{20}$$

In the sequel, we show that the function u given in (18) is a solution of the problem $(\mathcal{P}_1)$ only if $\mathcal{A}$ and $\mathcal{B}$ are as in (19) and (20), respectively. Let

$$u(t) = \int_0^1 G(t,s)v(s)\,ds + \mathcal{A}\,x(t) + \mathcal{B}\,y(t), \quad t\in[0,1], \tag{21}$$

be a solution of the problem $(\mathcal{P}_1)$, then we have that $\forall t \in [0,1]$

$$u(t) = \int_0^t \frac{1}{\rho}x(s)y(t)v(s)\,ds + \int_t^1 \frac{1}{\rho}x(t)y(s)v(s)\,ds + \mathcal{A}\,x(t) + \mathcal{B}\,y(t),$$

$$p(t)u'(t) = p(t)y'(t)\int_0^t \frac{1}{\rho}x(s)v(s)\,ds + p(t)x'(t)\int_t^1 \frac{1}{\rho}y(s)v(s)\,ds + \mathcal{A}\,p(t)x'(t) + \mathcal{B}\,p(t)y'(t),$$

and

$$\begin{aligned}(p(t)u'(t))' = \;& (p(t)y'(t))' \int_0^t \tfrac{1}{\rho}x(s)v(s)\,ds + p(t)y'(t)\tfrac{1}{\rho}x(t)v(t) \\ & +(p(t)x'(t))' \int_t^1 \tfrac{1}{\rho}y(s)v(s)\,ds - p(t)x'(t)\tfrac{1}{\rho}y(t)v(t) \\ & +\mathcal{A}\,(p(t)x'(t))' + \mathcal{B}\,(p(t)y'(t))',\end{aligned}$$

so that

$$\begin{aligned}(p(t)u'(t))' - q(t)u(t) = \;& \tfrac{p(t)}{\rho}\left(y'(t)x(t) - x'(t)y(t)\right)v(t) \\ & ((p(t)y'(t))' - q(t)y(t)) \int_0^t \tfrac{1}{\rho}x(s)v(s)\,ds \\ & + ((p(t)y'(t))' - q(t)y(t)) \int_t^1 \tfrac{1}{\rho}y(s)v(s)\,ds \\ & +\mathcal{A}\,((p(t)y'(t))' - q(t)y(t)) + \mathcal{B}\,((p(t)y'(t))' - q(t)y(t)) \\ & = -\tfrac{p(t)}{\rho}\left(-y'(t)x(t) + x'(t)y(t)\right)v(t) \\ & = -\tfrac{p(t)}{\rho}\begin{vmatrix} y(t) & x(t) \\ y'(t) & x'(t)\end{vmatrix} v(t) \\ & = -\tfrac{p(t)}{\rho}v(t)\begin{vmatrix} y(0) & x(0) \\ y'(0) & x'(0)\end{vmatrix}\tfrac{p(0)}{p(t)} \\ & = -v(t).\end{aligned}$$

Since

$$u(0) = x(0)\int_0^1 \frac{1}{\rho} y(s)v(s)\,ds + \mathcal{A}\,x(0) + \mathcal{B}\,y(0),$$
$$u'(0) = x'(0)\int_0^1 \frac{1}{\rho} y(s)v(s)\,ds + \mathcal{A}\,x'(0) + \mathcal{B}\,y'(0),$$

then

$$\mathcal{B}\left(ay(0) - by'(0)\right) = \sum_{i=1}^{m-2} a_i \left[\int_0^1 G(\xi_i, s)v(s)\,ds + \mathcal{A}\,x(\xi_i) + \mathcal{B}\,y(\xi_i)\right]. \tag{22}$$

Since

$$u(1) = y(1)\int_0^1 \frac{1}{\rho} x(s)v(s)\,ds + \mathcal{A}\,x(1) + \mathcal{B}\,y(1),$$
$$u'(1) = x'(1)\int_0^1 \frac{1}{\rho} x(s)v(s)\,ds + \mathcal{A}\,x'(1) + \mathcal{B}\,y'(1),$$

then

$$\mathcal{A}\left(cx(1) + dx'(1)\right) = \sum_{i=1}^{m-2} b_i \left[\int_0^1 G(\xi_i, s)v(s)\,ds + \mathcal{A}\,x(\xi_i) + \mathcal{B}\,y(\xi_i)\right]. \tag{23}$$

From (22) and (23), we obtain that

$$\begin{cases} \left[-\sum\limits_{i=1}^{m-2} a_i x(\xi_i)\right]\mathcal{A} + \left[\rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i)\right]\mathcal{B} &= \sum\limits_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)v(s)\,ds \\ \left[\rho - \sum\limits_{i=1}^{m-2} b_i x(\xi_i)\right]\mathcal{A} - \left[\sum\limits_{i=1}^{m-2} b_i y(\xi_i)\right]\mathcal{B} &= \sum\limits_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s)v(s)\,ds \end{cases}$$

From the first equation we have

$$\mathcal{A} = \frac{\sum\limits_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)v(s)\,ds - \left[\rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i)\right]\mathcal{B}}{\left[-\sum\limits_{i=1}^{m-2} a_i x(\xi_i)\right]} \tag{24}$$

we replace in the second equation

$$\left[\rho - \sum_{i=1}^{m-2} b_i x(\xi_i)\right]\left[\frac{\sum\limits_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)v(s)\,ds - \left[\rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i)\right]\mathcal{B}}{\left[-\sum\limits_{i=1}^{m-2} a_i x(\xi_i)\right]}\right] - \left[\sum_{i=1}^{m-2} b_i y(\xi_i)\right]\mathcal{B} = \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s)v(s)\,ds$$

so,

$$\begin{aligned} &\left[\rho - \sum_{i=1}^{m-2} b_i x(\xi_i)\right]\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)v(s)\,ds - \left[\rho - \sum_{i=1}^{m-2} b_i x(\xi_i)\right]\left[\rho - \sum_{i=1}^{m-2} a_i y(\xi_i)\right]\mathcal{B} \\ &- \left[-\sum_{i=1}^{m-2} a_i x(\xi_i)\right]\left[\sum_{i=1}^{m-2} b_i y(\xi_i)\right]\mathcal{B} = \left[-\sum_{i=1}^{m-2} a_i x(\xi_i)\right]\sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s)v(s)\,ds \end{aligned}$$

then,

$$\begin{aligned} &\left[\rho - \sum_{i=1}^{m-2} b_i x(\xi_i)\right]\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)v(s)\,ds - \left[-\sum_{i=1}^{m-2} a_i x(\xi_i)\right]\sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s)v(s)\,ds \\ &= \mathcal{B}\left(\left[\rho - \sum_{i=1}^{m-2} b_i x(\xi_i)\right]\left[\rho - \sum_{i=1}^{m-2} a_i y(\xi_i)\right] - \left[\sum_{i=1}^{m-2} a_i x(\xi_i)\right]\left[\sum_{i=1}^{m-2} b_i y(\xi_i)\right]\right) \end{aligned}$$

then

$$
\begin{aligned}
&-\left[\rho-\sum_{i=1}^{m-2} b_i x(\xi_i)\right]\sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)v(s)\,ds+\left[-\sum_{i=1}^{m-2} a_i x(\xi_i)\right]\sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)v(s)\,ds\\
&=\mathcal{B}\left(-\left[\rho-\sum_{i=1}^{m-2} b_i x(\xi_i)\right]\left[\rho-\sum_{i=1}^{m-2} a_i y(\xi_i)\right]+\left[-\sum_{i=1}^{m-2} a_i x(\xi_i)\right]\left[-\sum_{i=1}^{m-2} b_i y(\xi_i)\right]\right)
\end{aligned}
$$

this is implies that $\mathcal{B}$ and satisfy (20), and replacing $\mathcal{B}$ in (24) we obtain that $\mathcal{B}$ is as (20). We can easily verify that the function u given in (18) is a solution of the problem $(\mathcal{P}_1)$ if $\mathcal{A}$ and $\mathcal{B}$ are defined by (19) and (20), respectively.

II. Fourth order differential equation at two points

Consider the following fourth order differential equation at two point

$$u^{(4)}(t)=\phi(t),\quad t\in[0,1], \tag{25}$$

with two point boundary conditions

$$u(0)=u(1)=u''(0)=u''(1)=0. \tag{26}$$

We set $v=u''$, then the problem is reduced to the following second order problems with Dirichlet boundary conditions

$$
(\mathcal{A}_1)\begin{cases} v''(t)=\phi(t) & 0<t<1,\\ v(0)=v(1)=0. \end{cases}\qquad
(\mathcal{A}_2)\begin{cases} u''(t)=v(t) & 0<t<1,\\ u(0)=u(1)=0. \end{cases}
$$

Clearly the solutions v and u of the above problems depend on ϕ.

The solution of the problems $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ can be represented respectively in the forme

$$v(x)=-\int_0^1 G_1(x,t)\phi(t)\,dt,$$

$$u(x)=-\int_0^1 G_1(x,t)v(t)\,dt,$$

where $G_1(x,t):[0,1]\times[0,1]\to\mathbb{R}$ denotes the Green function for the differential operator $-u''$ with homogenous Dirichlet boundary condition

$$
G_1(x,t)=\begin{cases} x(1-t), & 0\le x\le t\le 1,\\ t(1-x), & 0\le t\le x\le 1. \end{cases}
$$

Then

$$\begin{aligned} u(t) = -\textstyle\int_0^1 G_1(x,t)v(t)\,dt \quad &= -\textstyle\int_0^1 G_1(x,t)\left(-\textstyle\int_0^1 G_1(t,s)\phi(s)\,ds\right)\,dt \\ &= \textstyle\int_0^1\int_0^1 G_1(x,t)G_1(t,s)\phi(s)\,ds\,dt \\ &= \textstyle\int_0^1\left(\int_0^1 G_1(x,t)G_1(t,s)\,dt\right)\phi(s)\,ds \\ &= \textstyle\int_0^1 G(t,s)\phi(s)\,ds \end{aligned}$$

where $G:[0,1]\times[0,1]\to\mathbb{R}$

$$G(t,s)=\int_0^1 G_1(x,t)G_1(t,s)\,dt=\begin{cases} t(1-s)\frac{2s-s^2-t^2}{6}, & 0\le t\le s\le 1,\\ s(1-t)\frac{2t-t^2-s^2}{6}, & 0\le s\le t\le 1.\end{cases}$$

General conclusion

This thesis is a contribution dedicated to the fixed point theory for the sum of two operators on cones and its applications. Our work was motivated, in one hand, by the fact that many problems arising in applied sciences can be formulated as fixed point equation for the sum of two operators in appropriate spaces. In the other hand, it is motivated by the fact that the positivity of a solutions which may represent a density, a temperature, a velocity, a gravity, and more, is a very important issue in applications.

In this thesis, several fixed point theorems of Krasnosel'skii type and Leggett-Williams type are extended to the class of mappings of the form $T + S$, where $(I - T)$ is Lipschitz invertible and S is a k-set contraction. The arguments are based upon recent fixed point index theory developed by Mebarki et al. in [34, 43]. The obtained results and other recent ones are used to investigate the existence of positive solutions for various classes of boundary value problems for difference and differential equations, and then new existence criteria are obtained. As prospects this new approach can be used to study other kinds of boundary value problems for differential, partial differential, fractional and differences equations, and even for dynamic equations on times scales. This approach can also adopted in:

- Metric fixed point theory.
- Random fixed point theory.
- Discrete fixed point theory (Tarski's fixed point theorem).

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ABSTRACT

In this work, we are concerned with the study of existence, multiplicity, positivity, and localization of solutions various kinds of boundary value problems for difference and differential equations. The approach used is the fixed point theory for the sum of two operators on the retracts of Banach spaces. Firstly, we developed new fixed point results for a class of k-set contractions perturbed by an operator T such that $(I - T)$ is Lipschitz invertible on cones as well as on the translates of cones. Then, we used these results to obtain new criteria that ensures existence, multiplicity and localization of positive solutions for diverse classes of boundary value problems. Most of the obtained theoretical criteria are illustrated by numerical examples.

Key words : Fixed point ; sum of operators ; fixed point index ; cone ; Banach space ; boundary value problems ; difference equations ; differential equations.

RÉSUMÉ

Dans ce travail, nous nous intéressons à l'étude de l'existence, la multiplicité, la positivité et la localisation des solutions de divers types de problèmes aux limites associés à des équations aux différences et à des équations différentielles. L'approche utilisée est la théorie du point fixe pour la somme de deux opérateurs sur les rétractés des espaces de Banach. D'une part, nous avons développé de nouveaux théorèmes de points fixes pour une classe une k-contractions d'ensembles perturbées par un opérateur T tel que $(I - T)$ est Lipschitz inversible sur les cônes ainsi que sur les translations des cônes. D'autre part, nous avons utilisé ces résultats pour obtenir de nouveaux critères qui assurent l'existence, la multiplicité et la localisation de solutions positives pour différentes classes de problèmes aux limites. La plupart des critères théoriques obtenus sont illustrés par des exemples numériques.

Mots clés : Point fixe ; somme d'opérateurs ; indice de point fixe ; cône ; espace de Banach ; problèmes aux limites ; équations aux différences ; équations différentielles.

ملخص

نهتم في هذا العمل بدراسة وجود ، تعددية ، إجابية و تأطير الحلول لأنواع مختلفة من المسائل الحدية المرتبطة بمعادلات الفروق والمعادلات التفاضلية. النهج المستخدم هو نظرية النقطة الصامدة لمجموع مؤثرين في فضاءات باناخية. من ناحية ، قمنا بتطوير بعض نتائج النقطة الصامدة لمجموع مؤثرين معرفين على مخروط وعلى انسحاب مخروط في فضاء بناخي. من ناحية أخرى ، استخدمنا هذه النتائج للحصول على معايير جديدة تضمن وجود و تعدد الحلول الموجبة للمسائل الحدية المدروسة

الكلمات المفتاحية : النقطة الصامدة ; مجموع مؤثرين ; مؤشر النقطة الصامدة ; مخروط ; فضاء باناخ ; المسائل الحدية ; معادلات الفروق ; المعادلات التفاضلية.