République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
Université A. Mira de Bejaia

جامعة بجاية
Tasdawit n Bgayet
Université de Béjaïa

Université A. Mira de Bejaia

Département de Mathématiques

Laboratoire de Mathématiques Appliquées

Thèse

en vue de l'obtention du diplôme de Doctorat
Domaine : Mathématiques et Informatique Filière : Mathématiques
Spécialité : Analyse

Présentée par
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Thème

Contribution à la théorie du point fixe pour la somme de deux opérateurs et applications

Soutenue le : 18/06/2023 **Devant le jury composé de :**

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Année Universitaire : 2022/2023

People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
A. Mira University of Bejaia

Faculty of Exact Sciences

Department of Mathematics

Laboratory of Applied Mathematics

Thesis

In order to get a doctoral degree
Domain : Mathematics and Informatics Track : Mathematics
Option : Analysis

Presented by
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Theme

Contribution to the fixed point theory for the sum of two operators and applications

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Academic Year: 2022/2023

Remerciements

Tout d'abord, je tiens à remercier profondément mon directeur de thèse **Prof. KHELOUFI-MEBARKI Karima**, professeur à l'université de Bejaia, de m'avoir proposé ce sujet et qui m'a encadré tout au long de ce travail de recherche. Elle n'a pas hésité à m'encourager. Elle m'a fait partagé ses brillantes intuitions. C'est grâce à elle que ce travail a été fait. Qu'elle soit aussi remerciée pour sa disponibilité permanente, son aide, ses conseils et son encadrement au cours de l'élaboration de ce travail.
Mes remerciements vont également au **Prof. BOURAINE-BERDJOUDJ Louiza**, professeur à l'université de Bejaia, de m'avoir fait l'honneur de présider le jury de cette thèse. Je remercie également, **Dr. HALIM Yacine**, Maître de conférences A au centre universitaire de Mila et **Dr. BOUKOUCHA Rachid**, Maître de conférences A à l'université de Bejaia d'avoir accépté d'examiner ce travail.
Merci aussi à **Dr. ALLILI-ZAHAR Samira**, Maître de conférences B à l'université de Bejaia d'avoir accepté notre invitation pour assister à ma soutenance. Je remercie également **Prof. Svetlin Georgiev**, de l'université de Sofia pour sa collaboration.

Mes derniers remerciements vont à tous les membres du **Laboratoire de Mathématiques Appliquées** et aux enseignants du département de mathématiques de l'université de Bejaia, notamment ceux avec qui j'ai eu l'occasion de travailler.

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Abstract

The main objective of this thesis is the study of some questions related to the existence of fixed points for the sum of two operators defined on ordered Banach spaces. The essential questions of this work are existence, positivity, localization and multiplicity of solutions for nonlinear equations which are written in the form $Tx + Fx = x$, where $(I - T)$ is a Lipchitz invertible mapping with constant $\gamma > 0$ and F is a k-set contraction with $k\gamma < 1$. Note that many mathematical problems, related to nonlinear differential or integral equations, can be written in the previous form. At first, we present some preliminary elements and results for the elaboration of this thesis such as Kuratowski measure of noncompactness and the fixed point index theory on cones. Secondly, we develop a new fixed point theorem for sum of two operators, using the fixed point index theory. This theory provides practical techniques for establishing fixed point theorems in ordered Banach spaces. Finally, the obtained results will be applied to the study of certain types of boundary value problems associated to ODEs.

Keywords: Fixed point index, cone, sum of operators, Green's function, nonnegative solution, ODE, first-order boundary value problems, three-point boundary value problem.

Résumé

Le but principal de cette thèse est l'étude de quelques questions liées à l'existence de points fixes pour la somme de deux opérateurs définis sur un espace de Banach ordonné. Les questions essentielles de ce travail sont l'existence, la positivité, la localisation et la multiplicité de solutions pour des équations différentielles ou intégrales non linéaires qui s'écrivent sous la forme $Tx + Fx = x$, où $(I - T)$ est un opérteur dont l'inverse est lipchitzien de constante $\gamma > 0$ et F est une k-contraction d'ensembles avec $k\gamma < 1$. Notons que de nombreux problèmes mathématiques, liés à des équations différentielles ou intégrales non linéaires, peuvent être écrits sous la forme précédente. En premier lieu, nous présentons quelques éléments et résultats préliminaires pour l'élaboration de cette thèse, notamment la mesure de non-compacité de Kuratowski et la théorie de l'indice du point fixe. En deuxième lieu, nous développons un nouveau théorème du point fixe pour la somme de deux opérateurs, en utilisant la théorie de l'indice du point fixe les cônes. Une théorie qui fournit des techniques pratiques pour établir des théorèmes de points fixes dans des espaces de Banach ordonnés. Enfin, les résultats obtenus seront appliqués à l'étude de certains types de problèmes aux limites associés à des EDOs.

Mots-clés : Indice de point fixe, cône, somme d'opérateurs, fonction de Green, EDO, solution positive, problèmes aux limites d'ordre 1, problème aux limites à trois points.

List of publications

- ***A. Mouhous, K. Mebarki.*** *Existence of fixed points in conical shells of a Banach space for sum of two operators and application in ODEs,* **Turkish Journal of Mathematics, Vol. 46, No. 7, (2022), 2556-2571**.

- ***A. Mouhous, S.G. Georgiev, K. Mebarki.*** *Existence of Solutions for a Class of First Order Boundary Value Problems,* **Archivum Mathematicum, Vol. 58, No.3, (2022), 141–158.**

Introduction

The first objective of this thesis is the study of the existence of fixed points for certain classes of operators defined on cones of Banach spaces. More precisely, we are interested in questions related to the existence, nonnegativity, localization and multiplicity of fixed points for some operators that are of the form $T + F$ where $I - T$ is a Lipschitz invertible mapping and F is a k-set contraction. The approach used is the fixed point index theory on cones of Banach spaces. The second objective of this thesis is to study the existence and the multiplicity of nonnegative solutions for certain classes of ordinary differential equations subjected to different boundary conditions. The approach used is to reduce the study, under suitable conditions, to the existence of fixed points for appropriate operators. Recent fixed point theorems are used to show the existence of the fixed points of these operators which are solutions of our problems.

Many problems in science can be mathematically recast as nonlinear equations of the form $Tx + Fx = x$ and posed in some closed convex subset of a Banach space. Notice further that the nonnegativity of solutions of nonlinear equations, especially ordinary, fractional, partial differential equations, and integral equations is a very important issue in applications, where a nonnegative solution may represent a density, temperature, velocity, gravity, etc. It's the reason for which many recent research works investigate not only the existence but also the nonnegativity of solutions for various types of nonlinear equations.

Starting from the Krasnoselskii's fixed point theorem in cones [32, 33], appeared in 1960, the fixed point theory in cones developed promptly and has been widely extended to various directions in theory as well as in applications to many problems in nonlinear sciences. Very recently, in 2019, the authors in [16] open a new direction of research in the theory of fixed point in cones for the sum of two operators. Several fixed point theorems, including Krasnosel'skii's theorems type, have being established for a sum of an expansive operator and a k-set contraction. Recent developments of positive fixed point theorems, in this direction, and their applications can be found in [11, 14, 15, 16, 21, 22, 23, 24]. One of our contributions in this thesis is part of generalizations leading to fixed point theory for sums of two operators.

This thesis is organized as follows: The first chapter will be devoted to the general framework. Some preliminaries results and basic concepts used throughout this thesis are collected here. Section 1.1 of this chapter opens with cones in Banach spaces which is required in this study since it is the tool that provides the ordering needed to describe the nonnegativity of the solution. However, some compactness criteria and the classical Kuratowski measure of noncompactness (KMNC for short) of a set in a metric space occupies the major part of Section 1.1. The fixed point index is a generalization of the Leray-Schauder degree. Section 1.2 starts with a reminder of the main properties of the fixed point index for strict set contractions defined in bounded convex subsets of Banach spaces, in particular on cones. The definition of a generalized fixed point index for k-set contraction perturbed by an expansive mapping as well as some of its properties are presented. The case of a k-set contraction perturbed by any mapping T such that $(I-T)$ is Lipschitz invertible is also discussed in this section. As a consequence, some fixed point theorems for the sum of two operators are derived in Section 1.3.

In Chapter 2, the integral formulation of all boundary value problems studied in this thesis are presented in details.

Our contributions in this thesis are presented in chapters 3 and 4.

Chapter 3 is devoted to investigate the existence of solutions to the following first order differential equation

$$x' = f(t, x), \quad t \in [a, b],$$

subject to the boundary conditions

$$Mx(a) + Rx(b) = 0,$$

where $M, R \in \mathbb{R}$, $M + R \neq 0$, $a < b < \infty$ are given constants and

(H1) $f \in \mathcal{C}([a, b] \times \mathbb{R})$, $|f(t, x)| \leq \sum_{j=1}^{k} a_j(t)|x|^{p_j}$, $(t, x) \in [a, b] \times \mathbb{R}$, $a_j \in \mathcal{C}([a, b])$, $0 \leq a_j \leq A$ on $[a, b]$, $p_j \geq 0$, $j \in \{1, \ldots, k\}$.

Under sufficient conditions, we show that the considered problem two nontrivial nonnegative solutions. The results of this chapter vary according to the hypotheses on the nonlinear term of the studied differential equation. To prove our main results we propose a new approach based upon recent theoretical results. The proof of our results made use of two recent fixed point theorems for the sum of two operators presented in Section 1.3. Noting that, we can consider our main results obtained in this chapter as complementary ones to these, of the scalar-valued case, established in [44].

In Chapter 4, the functional expansion-compression fixed point theorem of Leggett-Williams type developed in [3] is extended to the class of mappings of the form $T + F$, where $(I - T)$ is Lipschitz invertible and F is a k-set contraction. The arguments are based upon recent fixed point index theory for this class of mappings. As application, our approach is applied to prove the existence of nontrivial nonnegative solutions for the three-point boundary value problem:

$$\begin{aligned} &y'' + f(t,y) = 0, \quad t \in (0,1), \\ &y(0) = ky(\eta), \quad y(1) = 0, \end{aligned} \tag{0.0.1}$$

where $\eta \in (0,1)$, $k > 0$ with $k(1-\eta) < 1$ and $f \in \mathcal{C}([0,1] \times [0,\infty))$.

We show an existence criterion under the assumptions:

($\mathcal{C}$1) $\tilde{A} < f(t,y) \leq a_1(t) + a_2(t)|y|^p$ for $t \in [0,1]$ and $y \in [0,\infty)$, $a_1, a_2 \in \mathcal{C}([0,1])$, $0 \leq a_1, a_2 \leq A$ on $[0,1]$, for some positive constants $A, \tilde{A}$ and p.

($\mathcal{C}$2) $\epsilon \in (0,1)$, and there exist $a, b, c, d, z_0, \rho > 0$ such that

$$\begin{aligned} &\max(d, \tfrac{2z_0}{\epsilon}, \tfrac{1}{\Lambda}(c - z_0)) < b \leq \rho; \\ &3z_0 > a; \quad z_0 \leq c < \min(a, 3\,z_0, \tfrac{\eta}{3}\left(1 - \tfrac{\eta}{2}\right)\tilde{A} + (1 - \tfrac{1}{\epsilon})z_0); \\ &\tfrac{\epsilon AB(1+b^p)+3z_0}{\epsilon} \leq \rho; \quad (1-\epsilon)\tfrac{c}{\Lambda} + 3z_0 \leq d, \quad \text{where } \Lambda = \frac{\min\left(\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right)}{\epsilon\,\rho}, \\ &AB(1 + b^p) < b, \quad \text{where } B = \tfrac{1+k\eta}{1-k(1-\eta)}. \end{aligned}$$

The thesis ends by a general conclusion.

Notations

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of all nonnegative real numbers.
$\mathbb{R}^n$	The n-dimentional Euclidean space.
(a, b)	Open interval $]a, b[$.
$\inf(A)$	The infimum of the set A.
$\sup(A)$	The supremum of the set A.
$\partial\Omega$	The boundary of Ω.
$\mathring{\Omega}$	The interior of Ω.
$\overline{\Omega}$	Adhesion de Ω.
$\|.\|$	A norm.
I	The identity application.
$BVPs$	Boundary value problems.
$f\|_V$	The restriction of f on V.
$i(f, U, D)$	Fixed point index of f on U with respect to D.
$Fix(f)$	The set of fixed points of f.
$\mathcal{P}$	Cone.
$\mathcal{P}^*$	$\mathcal{P}/\{0\}$.
$\mathcal{C}(\Omega)$	The set of all real continuous functions from Ω in $\mathbb{R}$.
$mes(D)$	The Lebegues measure of the set D.
$\mathcal{P}(\psi, R)$	$\{x \in \mathcal{P} : \psi(x) \leq R\}$, where ψ be a nonnegative continuous functionals on $\mathcal{P}$.

1 Fixed point theory on cones

1.1 Basic Concepts

1.1.1 Cones in Banach spaces

In this chapter, we will collect some notations, definitions and auxiliary results we need throughout this thesis. Let E a Banach space.

Definition 1.1.1. *A subset $\mathcal{P} \subset E$ is called cone if the following conditions are satisfied:*

1. *$\mathcal{P}$ is closed, convex and $\mathcal{P} \neq \emptyset$.*

2. *If $(x \in \mathcal{P}$ and $\lambda \geq 0)$ then $\lambda x \in \mathcal{P}$.*

3. *If $(x \in \mathcal{P}$ and $-x \in \mathcal{P})$ then $x = 0$, i.e., $(\mathcal{P} \cap (-\mathcal{P}) = \{0\})$.*

Definition 1.1.2. *For any cone $\mathcal{P}$ in E, we can define a partial order relation $\leq$ on E as follows: $\forall x, y \in E : \; x \leq y \Leftrightarrow y - x \in \mathcal{P}$.*

We can also define the following partial order relations:

- $x < y \Leftrightarrow x \leq y$ and $x \neq y$.

- $x \ll y \Leftrightarrow y - x \in \mathring{\mathcal{P}}$ if $\mathring{\mathcal{P}} \neq \emptyset$.

- $x \nleq y \Leftrightarrow y - x \notin \mathcal{P}$.

Definition 1.1.3. *A segment of a cone $\mathcal{P}$ is defined by:*

$$[x, y] = \{z \in \mathcal{P} : \ x \leq z \leq y\}.$$

Definition 1.1.4. *Let $\mathcal{P}$ a cone in E.*

- *$\mathcal{P}$ is called normal if there exists a positive constant δ such that*

$$\|x + y\| \geq \delta, \ \forall x, y \in \mathcal{P} \ \textit{with} \ \|x\| = \|y\| = 1.$$

  Remark 1.1.5. *Geometrically, the normality of a cone means that the angle between any two positive unit vectors cannot exceed π. In other words, a normal cone can not be too large.*

- *$\mathcal{P}$ is called solid if $\mathring{\mathcal{P}} \neq \emptyset$, where $\mathring{\mathcal{P}}$ is the interior of $\mathcal{P}$.*

- *$\mathcal{P}$ is generator if $E = \mathcal{P} - \mathcal{P}$, i.e, $\forall x \in E, \exists\ u, v \in \mathcal{P}$ such that: $x = u - v$ (In other words any element $x \in E$ can be written in the form: $x = u - v$ where $u, v \in \mathcal{P}$).*

The following theorem gives us other definitions of a normal cone.

Theorem 1.1.6. ([27, Theorem 1.1.1]). *Let $\mathcal{P}$ a cone of a Banach space E. Then the following assertions are equivalent:*

1. *$\mathcal{P}$ is normal;*

2. *$\exists \gamma > 0$ such that $\|x + y\| \geq \gamma \max(\|x\|, \|y\|), \ \forall x, y \in \mathcal{P}$;*

3. *$\exists N > 0$ such that $0 \leq x \leq y \Rightarrow \|x\| \leq N \|y\|, \ \forall x, y \in \mathcal{P}$;*

   (i.e. the norm $\|.\|$ is semi monotone).

4. *there exist an equivalent norm* $\|.\|_1$ *on* E *such that:* $0 \leq x \leq y \Rightarrow \|x\|_1 \leq \|y\|_1,\ \forall x, y \in \mathcal{P}$; *(i.e. the norm* $\|.\|_1$ *is monotone);*

5. *any ordered interval* $[x, y] = \{z \in E :\ x \leq z \leq y\}$ *is bounded.*

Example 1.1.7.

1. *Let* $E = \mathbb{R}^n$ *and* $\mathcal{P}_1 = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n :\ x_i \geq 0,\ i = 1, ..., n\} = (\mathbb{R}_+)^n$.

   (a) $\mathcal{P}_1$ *is a solid and generator cone in* $\mathbb{R}^n$, *because* $\mathring{\mathcal{P}}_1 = (\mathbb{R}_+^*)^n$ *and since* $\mathbb{R}_+$ *is a generator cone on* $\mathbb{R}$, *then for* $i = 1, ..., n : \forall x_i \in \mathbb{R}, \exists u_i, v_i \in \mathbb{R}_+ : x_i = u_i - v_i$.

   (b) *Since every norm on* $\mathbb{R}^n$ *is monotone, we have*

   $$\forall x, y \in \mathbb{R}^n,\ 0_{\mathbb{R}^n} \leq x \leq y \Rightarrow \|x\| \leq \|y\|.$$

   So $\mathcal{P}_1$ *is normal with the normality constant* $N = 1$.

2. *Let* $E = \mathcal{C}(G)$, *the space of continuous functions on a bounded set* $G \subset \mathbb{R}^n$ *be endowed with the maximum norm* $\|x\|_{\mathcal{C}(G)} = \sup_{t \in G} |x(t)|$, *and*

   $$\mathcal{P}_2 = \{x \in \mathcal{C}(G) : x(t) \geq 0,\ \forall t \in G\}.$$

   (a) $\mathcal{P}_2$ *is a solid cone and generator on* $\mathcal{C}(G)$.

   (b) $\mathcal{P}_2$ *is normal because the norm* $\|.\|_{\mathcal{C}(G)}$ *is monotone on* $\mathcal{C}(G)$.

   (c) *We can define other solid normal cones on* $\mathcal{C}(G)$ *such that:*

   $$\mathcal{P}_3 = \{x \in \mathcal{C}(G) : x(t) \geq 0,\ \text{and}\ \min_{t \in G_1} x(t) \geq \varepsilon_1 \|x(t)\|_{\mathcal{C}(G)}\},$$

   where G_1 *is closed subsets of* G *and* $\varepsilon_1 \in (0, 1)$.

1.1.2 Some compactness criteria

Ascoli-Arzelà criterion

Definition 1.1.8 (**Equicontinuous set**)**.** *Let (X,τ) be a topological space, (X,d) a metric space, and $\mathcal{C}(X,Y)$ denotes the space of continuous functions from X to Y.*
$\mathcal{A}\subset\mathcal{C}(X,Y)$ is called equicontinuous at a point $x_0\in X$ if and only if

$$\forall\,\epsilon>0,\ \exists U_\epsilon\in\mathcal{V}(x_0),\ \forall f\in\mathcal{A},\ \forall x\in X\ \ (x\in U_\epsilon\Longrightarrow f(x)\in B(f(x_0),\epsilon))\,.$$

The set $\mathcal{A}$ is equicontinuous if it is equicontinuous at every point $x_0\in X$.

Remark 1.1.9. *If (X,d) is a compact metric space, then $\mathcal{A}$ is equicontinuous if $\forall\epsilon>0$, $\exists\eta>0$, $\forall x,y\in X\ \ (d(x,y)<\eta\Longrightarrow d(f(x),f(y))<\epsilon)\,,\ \forall f\in\mathcal{A}$.*

To prove Ascoli-Arzelà Theorem we consider, for the sake of simplicity, a special situation in which (X,d) is a compact metric space and $(Y,\|\cdot\|)$ a Banach space. The space $E=\mathcal{C}(X,Y)$ is endowed with the norm: $\|f\|=\sup\limits_{x\in X}\|f(x)\|_Y$.

Theorem 1.1.10 (Ascoli-Arzelà Theorem)**.** *A subset $\mathcal{H}\subset\mathcal{C}(X,Y)$ is relatively compact if and only if*

(a) $\mathcal{H}$ is equicontinuous.

(b) $\forall\,x\in X$, the set $\mathcal{H}(x)=\{f(x),\ f\in\mathcal{H}\}$ is relatively compact in Y.

Proof.

(1) The condition is necessary. If $\mathcal{H}$ is relatively compact, then for all $\varepsilon>0$, there exist a finite number of elements $\{f_i\}_{1\le i\le n}$ in E such that $\mathcal{H}\subset\bigcup\limits_{i=1}^{n}B(f_i,\varepsilon/3)$, i.e.,

$$\forall\,f\in\mathcal{H},\exists\,i\in\{1,\ldots,n\},\ \|f-f_i\|_E\le\varepsilon/3.$$

Hence, $\forall\, f \in \mathcal{H}, \forall\, x \in X, \exists\, i \in \{1,\dots,n\},\ \|f(x) - f_i(x)\| \leq \varepsilon/3$.

Therefore, $\mathcal{H}(x) \subset \bigcup_{i=1}^{n} B\left(f_i(x), \varepsilon/3\right)$, which implies that $\mathcal{H}(x)$ is relatively compact in Y. Now we prove that $\mathcal{H}$ is equicontinuous. For all $i = 1, 2, \dots, n$, the function f_i is continuous. Then for all positive ε, there exists $\delta_i > 0$, such that $\forall\, x, y \in X$, we have

$$d(x, y) \leq \delta_i \Longrightarrow \|f_i(x) - f_i(y)\| \leq \varepsilon/3.$$

Let $\delta = \min\limits_{1\leq i\leq n} \delta_i$ and $f \in \mathcal{H}$. There exists $i \in \{1, \dots, n\}$ such that $f \in B(f_i, \varepsilon/3)$ and for all $x, y \in X$ we have

$$\begin{aligned} d(x,y) \leq \delta \Longrightarrow \|f(x) - f(y)\| &\leq \|f(x) - f_i(x)\| + \|f(y) - f_i(y)\| \\ &\quad + \|f_i(x) - f_i(y)\| \\ &\leq \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus the equicontinuity of $\mathcal{H}$.

(2) The condition is sufficient. Since $E = \mathcal{C}(X, Y)$ is complete, it is sufficient to prove that $\mathcal{H}$ is totally bounded. Let $\varepsilon > 0$. Since $\mathcal{H}$ is equicontinuous, for every $x \in X$, there exists some $\delta > 0$ such that for all $y \in X$ and $f \in \mathcal{H}$ we have

$$d(x, y) \leq \delta \Longrightarrow \|f(x) - f(y)\| \leq \varepsilon/4.$$

The space X being compact, can be covered by a finite number of balls $B_{x_i} = B(x_i, r),\ 1 \leq i \leq m$. By assumption, each subset $\mathcal{H}(x)$ is relatively compact in Y, then the same holds for their finite union $\mathcal{H} = \bigcup_{i=1}^{m} \mathcal{H}(x_i)$. Therefore, we can cover $\mathcal{H}$ by a finite number of balls centered at $c_j\, (1 \leq j \leq p)$ and with radius $\varepsilon/4$. Let $I = \{1, 2, \dots, m\},\ J = \{1, 2, \dots, p\}$, and let Φ be the set of all mappings $\varphi : I \longrightarrow J$. For all $\varphi \in \Phi$, denote by L_φ the set of all mappings $f \in \mathcal{H}$ such that $\forall\, i \in I,\ \|f(x_i) - c_{\varphi(i)}\| \leq \varepsilon/4$. Some of the sets L_φ may be empty, but $\mathcal{H}$ is covered by the union of L_φ. It remains to prove that the diameter of each L_φ is less than or equal to ε. Let $f, g \in L_\varphi$. For every $y \in X$, there exists $i \in I$ such that $y \in B_{x_i}$. Hence, $\|f(y) - f(x_i)\| \leq \varepsilon/4$ and $\|g(y) - g(x_i)\| \leq \varepsilon/4$.

So, for all $y \in Y$, we have

$$\begin{aligned} \|f(y) - g(y)\| &\leq \|f(y) - f(x_i)\| + \|g(y) - g(x_i)\| \\ &\quad + \|f(x_i) - c_{\varphi(i)}\| + \|g(x_i) - c_{\varphi(i)}\| \\ &\leq \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Hence, $\|f - g\| \leq \varepsilon$, and our claim follows. □

Consequently, when Y is finite dimensional, we get the following results.

Corollary 1.1.11. *If $\mathcal{H} \subset \mathcal{C}(X, Y)$ is uniformly bounded and equicontinuous, then $\mathcal{H}$ is relatively compact.*

Corollary 1.1.12. *Let $\mathcal{M} \subset \mathcal{C}^1([a, b], \mathbb{R})$ satisfy the following conditions:*

(a) there exists $L > 0$ such that for all $t \in [a, b]$ and $u \in \mathcal{M}$,

$$|u(t)| \leq L \quad \text{and} \quad |u'(t)| \leq L.$$

(b) For every positive real number $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $t_1, t_2 \in [a, b]$ with $|t_1 - t_2| < \delta(\varepsilon)$ and for all $u \in \mathcal{M}$,

$$|u(t_1) - u(t_2)| \leq \varepsilon \quad \text{and} \quad |u'(t_1) - u'(t_2)| \leq \varepsilon.$$

Then, the set $\mathcal{M}$ is relatively compact in $\mathcal{C}^1([a, b], \mathbb{R})$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{M} \subset \mathcal{C}^1([a, b], \mathbb{R})$. To prove that $\mathcal{M}$ is relatively compact in $\mathcal{C}^1([a, b], \mathbb{R})$, it is equivalent to show that $\{u_n\}_{n\in\mathbb{N}}$ has a subsequent converging in $\mathcal{C}^1([a, b], \mathbb{R})$. Since $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{M} \subset \mathcal{C}^1([a, b], \mathbb{R})$, $\{u'_n\}_{n\in\mathbb{N}}$ (resp. $\{u_n\}_{n\in\mathbb{N}}$) is a sequence of $\mathcal{C}([a, b], \mathbb{R})$. Corollary 1.1.11 and Assumptions (a)-(b) guarantee that the sequence of derivatives $\{u'_n\}_{n\in\mathbb{N}}$ (resp. $\{u_n\}_{n\in\mathbb{N}}$) is relatively compact in $\mathcal{C}([a, b], \mathbb{R})$.

As a consequence, there exists a subsequence, also denoted $\{u_n\}_{n\in\mathbb{N}}$ which converges in $\mathcal{C}([a,b],\mathbb{R})$ to a limit $u \in \mathcal{C}([a,b],\mathbb{R})$, and a subsequence of $\{u'_n\}_{n\in\mathbb{N}}$, also denoted $\{u'_n\}_{n\in\mathbb{N}}$, converging in $\mathcal{C}([a,b],\mathbb{R})$ to a limit $v \in \mathcal{C}([a,b],\mathbb{R})$. Using the integral representation of u_n, we find that for all $t, t_0 \in [a,b]$,

$$\begin{aligned} u_n(t) &= u(t_0) + \int_{t_0}^{t} u'_n(s)ds \\ &\to u(t_0) + \int_{t_0}^{t} v(s)ds, \quad \text{as } n \to \infty. \end{aligned}$$

Then for all $t \in [a,b]$, $\lim\limits_{n\longrightarrow\infty} u_n(t) = u(t)$ and the uniqueness of the limit yields that $u(t) = u(t_0) + \int_{t_0}^{t} v(s)ds$. Hence $u \in \mathcal{C}^1([a,b],\mathbb{R})$ and $u' = v$. □

In practice, the following result is widely used to study the compactness of a subset of $\mathcal{C}^k([a,b],\mathbb{R}^n)$.

Proposition 1.1.13. *For all $k \in \mathbb{N}$, the space $\mathcal{C}^{k+1}([a,b],\mathbb{R}^n)$ is embedded compactly in $\mathcal{C}^k([a,b],\mathbb{R}^n)$. Here "embedded compactly" means that every uniformly bounded sequence in $\mathcal{C}^{k+1}([a,b],\mathbb{R}^n)$ has a convergent subsequence in $\mathcal{C}^k([a,b],\mathbb{R}^n)$.*

Remark 1.1.14. *One of the aims of modern analysis is to characterize the relationship between various spaces of functions, An especially important type of relationship between Banach spaces is compact embedding: we say $E_1 \subset E_2$ is compactly embedded in E_2 if all bounded subsets of E_1 are relatively compact subsets of E_2. Compact embeddings provide a Bolzano-Weierstrass type theorem for infinite dimensions, since a sequence that is bounded in E_1 will contain a subsequence that converges strongly in E_2.*

Corduneanu-Avramescu Compactness Criterion

Let $\mathcal{C}_b([0,+\infty),\mathbb{R}^n)$ denote the vector topological space of all bounded and continuous functions defined on $[0,+\infty)$ and having values in $\mathbb{R}^n$. Before stating a compactness criterion in $\mathcal{C}_b([0,+\infty),\mathbb{R}^n)$, we give the definition of equi-convergent set.

Definition 1.1.15. *A subset $\mathcal{H} \subset \mathcal{C}_b([0,+\infty),\mathbb{R}^n)$ is called equi-convergent if*

$$\forall\,\varepsilon > 0,\ \exists\, T = T(\varepsilon) > 0,\ \forall\, t_1,\, t_2 \in \mathbb{R},$$

$$|t_1| > T,\ |t_2| > T\ \Rightarrow\ \|x(t_1) - x(t_2)\| < \varepsilon,\ \forall\, x \in \mathcal{H}.$$

Now, we state and prove Corduneanu-Avramescu compactness criterion in $\mathcal{C}_b([0,+\infty),\mathbb{R}^n)$.

Theorem 1.1.16. *[7] A subset $\mathcal{H} \subset \mathcal{C}_b([0,+\infty,\mathbb{R}^n)$ is relatively compact if and only if the following conditions are satisfied:*

(a) $\mathcal{H}$ is uniformly bounded.

(b) $\mathcal{H}$ is equicontinuous on every compact interval of $[0,+\infty)$
(we say that $\mathcal{H}$ is almost equicontinuous).

(c) $\mathcal{H}$ is equi-convergent.

Example 1.1.17. (1) *Let $G : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ be a continuous function and $T : \mathcal{C}([a,b],\mathbb{R}) \longrightarrow \mathcal{C}([a,b],\mathbb{R})$ be the linear operator defined by:*

$$Tx(t) = \int_a^b G(t,s)x(s)ds.$$

Then for any bounded set B in $\mathcal{C}([a,b],\mathbb{R})$, $T(B)$ is relatively compact.

(2) *Set $f_n(x) = \sin(nx)$, $x \in [0,2\pi]$ and $\mathcal{H} = \{f_n(.) : n \in \mathbb{N}\}$. Then $\mathcal{H}$ is uniformly bounded. In fact, $\|f_n\|_\infty \le 1, \forall\, n \in \mathbb{N}$. However, it is not equicontinuous in $\mathcal{C}([0,2\pi],\mathbb{R})$. In fact, consider the sequence $x_n = \frac{\pi}{2n}$, $n \in \mathbb{N}^*$, so $|f_n(x_n) - f_n(x_{2n})| = 1 - \frac{\sqrt{2}}{2} > \frac{1}{2}$. Hence, $\mathcal{H}$ is not relatively compact, i.e., we can't extract a convergent subsequence.*

1.1.3 Kuratowski's measure of non-compactness

We consider a real Banach space $(E, \|.\|)$ and let Ω_E be the class of all bounded subsets of E. In what follows, we will give the definition of Kuratowski non-compactness measure as well as its main properties. We will end this section by giving some examples. For more details on this concept, we refer to the references [8, 9, 10, 12].

Definition 1.1.18. *The Kuratowski's measure of non-compactness (KMNC for short) is the map* $\alpha : \Omega_E \longrightarrow [0, +\infty)$ *defined by :*

$$\alpha(A) = \inf \left\{ \begin{array}{c} d > 0 \ : \ A \text{ can be covered by finitely many sets} \\ \text{with diameter less than or equal } d \end{array} \right\},$$

that is to say

$$\alpha(A) = \inf \left\{ d > 0 \ : \ \exists A_1, ..., A_n \subset E, \ A \subseteq \bigcup_{i=1}^{n} A_i \text{ with } diam(A_i) \leq d, \ \forall i = 1, ..., n \right\},$$

where $diam(A_i) = \sup\limits_{x,y \in A_i} \|x - y\|$ *and* $diam(\emptyset) = 0$.

Remark 1.1.19.

1. *The definition of Kuratowski's measure of non-compactness is significant not only for Banach spaces but also for arbitrary metric spaces.*
2. $0 \leq \alpha(A) \leq diam(A) < \infty, \ \forall A \in \Omega_E$.
3. *A is finished* $\Longrightarrow \alpha(A) = 0, \ \forall A \in \Omega_E$.

Elementary properties of Kuratowski's non-compactness measure

The Kuratowski MNC α has the following properties (see [10, 12]).

Proposition 1.1.20. *Let A and B be bounded subsets of a Banach space E. Then the function* α *has the following properties:*

1. *Regularity:* $\alpha(A) = 0 \Longleftrightarrow \overline{A}$ *is compact.*

2. *Monotonicity:* $A \subset B \Longrightarrow \alpha(A) \leq \alpha(B)$ *i.e.,* α *is increasing.*

3. *Sub-additivity:* $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$.

4. $\alpha(A \cap B) \leq \min(\alpha(A), \alpha(B))$.

5. *Semi-homogeneity:* $\alpha(\lambda A) = |\lambda| \alpha(A), \forall \lambda \in \mathbb{R}$.

6. *Algebraic sub-additivity:* $\alpha(A+B) \leq \alpha(A) + \alpha(B)$.

7. *Invariance under passage to the closure:* $\alpha(A) = \alpha\left(\overline{A}\right)$.

8. *Invariance under shifting:* $\alpha(A+x) \leq \alpha(A), \forall x \in E$.

9. *Invariance under passage to the convex hull:* $\alpha(A) = \alpha(convA)$.

10. *Lipschitzianity:* $|\alpha(A) - \alpha(B)| \leq 2\mathcal{H}(A, B)$, *where* $\mathcal{H}(A, B)$ *denotes the Hausdorff distance between the sets* A *and* B.

Remark 1.1.21. *a) The properties of algebraic semi-homogeneity and sub-additivity allow us give that Kuratowski's measure of non-compactness* α *is a semi-norm on* E.

b) It's not easy to determine the explicit value of $\alpha(A)$ *for a bounded set* A *of* E. *Most of the results obtained using KMNC are based on its properties.*

Example 1.1.22.

1. *Let* $B(0,1)$ *a unit ball of a Banach space* E *of finite dimension, then* $\alpha(B(0,1)) = 0$. *Indeed, we have* $\overline{B(0,1)}$ *is compact* $\Longleftrightarrow \dim E < \infty$.
   More generally, $\alpha(B(x_0, r)) = 0$, *where* $B(x_0, r)$ *is a ball with center* x_0 *and radius* r *in a Banach space* E. *Indeed;* $\overline{B(x_0, r)}$ *is a compact of* E.

2. *Let* E *an infinite dimensional Banach space and* $B(0,1)$ *a unit ball of* E.
   Then $\alpha(B(0,1)) = 2$. *A simple proof is given in [12].*

3. *Let Y be a Banach space, and α the KMNC in Y. If $\mathcal{H} \subset E = \mathcal{C}([a,b], Y)$ is bounded and equicontinuous, then $\alpha_E(\mathcal{H}) = \max_{t\in[a,b]} \alpha(\mathcal{H}(t))$, where $\mathcal{H}(t) = \{x(t) : x \in \mathcal{H}\}$, $t \in [a,b]$. For the proof see [28, Theorem 1.2.4].*

4. *In the Banach space $E = \mathbb{R}^n$ with the Euclidean norm $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$, we consider the function $f = (f^{(1)}, \ldots, f^{(n)}) : E \to E$ defined by:*

$$f^{(i)}(t, x_1, \ldots, x_n, y_1, \ldots, y_n) \quad = \quad \tfrac{1-\cos t}{t+1}(1 + x_i^p + y_i^q), \quad \text{for } t \geq 0,$$

$x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $p, q \in (0, \infty)$, for any $i \in \{1, \ldots, n\}$.

Then, we have

$$\begin{aligned} \|f(t,x,y)\|^2 &= \sum_{i=1}^{n} (f^{(i)}(t,x,y))^2 \\ &\leq 4\tfrac{(1-\cos t)^2}{(t+1)^2}\left(n + \sum_{i=1}^{n} x_i^{2p} + \sum_{i=1}^{n} y_i^{2q}\right) \\ &\leq 4\tfrac{(1-\cos t)^2}{(t+1)^2}\left(n + (\sum_{i=1}^{n} x_i^2)^p + (\sum_{i=1}^{n} y_i^2)^q\right) \\ &\leq 4\tfrac{(1-\cos t)^2}{(t+1)^2}\left(n + \|x\|^{2p} + \|y\|^{2q}\right). \end{aligned}$$

Hence, let $D_1, D_2 \subset E$ bounded subset, for all $t \in \mathbb{R}^+$, $x \in D_1$, $y \in D_2$, we have

$$\|f(t,x,y)\| \leq 2\frac{1-\cos t}{t+1}\left(n + \|x\|^{2p} + \|y\|^{2q}\right) \leq 4\left(n + \|x\|^{2p} + \|y\|^{2q}\right) < \infty.$$

Moreover, for all $0 < t_1 < t_2 < +\infty$, $x \in D_1$ and $y \in D_2$, we get

$$\begin{aligned} &\lim_{t_1 \to t_2} |f^{(i)}(t_1,x,y) - f^{(i)}(t_2,x,y)| \\ \leq\ &\lim_{t_1 \to t_2} \left|\tfrac{1-\cos t_1}{t_1+1}(1 + x_i^p + y_i^q) - \tfrac{1-\cos t_2}{t_2+1}(1 + x_i^p + y_i^q)\right| \\ \leq\ &\lim_{t_1 \to t_2} (1 + \|x\|_\infty^p + \|x\|_\infty^q)\left|\tfrac{1-\cos t_1}{t_1+1} - \tfrac{1-\cos t_2}{t_2+1})\right| = 0, \ \forall\, i = 1, \ldots, n. \end{aligned}$$

Then, $\lim_{t_1 \to t_2} \|f(t_1,x,y) - f(t_2,x,y)\| = 0$ and

$$\begin{aligned} &\lim_{t \to +\infty} |f^{(i)}(t,x,y) - \lim_{s \to +\infty} f^{(i)}(s,x,y)| \\ \leq\ &\lim_{t \to +\infty} \left|\tfrac{1-\cos t}{t+1}(1 + x_i^p + y_i^q) - 0\right| = 0, \ \forall\, i = 1, \ldots, n. \end{aligned}$$

Hence, $\lim_{t\to+\infty} \|f^{(i)}(t,x,y) - \lim_{s\to+\infty} f^{(i)}(s,x,y)\| = 0.$

As a consequence, Corduneanu-Avramescu compactness criterion (see Theorem 1.1.16) ensures that $f(t, D_1, D_2)$ is relatively compact in $\mathbb{R}^n$. So, $\alpha(f(t, D_1, D_2)) = 0$, for all $t \in \mathbb{R}^+$ and all bounded subsets $D_1, D_2 \subset \mathbb{R}^n$.

1.1.4 Some classes of mappings: Definition and Properties

Nonnegative convex and nonnegative concave functionals

Definition 1.1.23. *Let $\mathcal{P}$ be a cone in a real Banach space E.*

(a) *A map $\chi : \mathcal{P} \longrightarrow [0,\infty)$ is said to be nonnegative convex functional on $\mathcal{P}$ if and only if $\chi(tx + (1-t)y) \leq t\chi(x) + (1-t)\chi(y)$, for all $x, y \in \mathcal{P}$ and $t \in [0,1]$.*

(b) *A map $\psi : \mathcal{P} \longrightarrow [0,\infty)$ is said to be nonnegative concave on $\mathcal{P}$ if and only if $\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$, for all $x, y \in \mathcal{P}$ and $t \in [0,1]$.*

Example 1.1.24. *(1) Let $\mathcal{P}$ be a cone in a real Banach space $(E, \|.\|)$.*
$\phi_1(x) = \|x\|$, $x \in \mathcal{P}$ is a convex functional.
(2) $\psi_1(x) = \min_{t\in[a,b]} x(t)$, $x \in \mathcal{C}([a,b], \mathbb{R}_+)$ is a concave functional.
(3) $\psi_2(x) = \int_{\Omega_1} x(t)\,dt$, $x \in \mathcal{C}(\Omega, \mathbb{R}_+)$, where Ω is a compact set of $\mathbb{R}^n$ and Ω_1 is a closed subset of Ω, is a concave functional.

I. Completely Continuous mappings

Let E and F be Banach spaces.

Definition 1.1.25. *Let $T : D \subset E \longrightarrow F$ be a continuous mapping. T is said to be:*

(i) *bounded if it maps any bounded subset of D into bounded subset of F;*

(ii) *compact if the set $T(D)$ is relatively compact;*

(iii) *completely continuous if it maps bounded sets into relatively compact sets.*

Remark 1.1.26 (Relation between compact and completely continuous maps)**.** *Every compact mapping is completely continuous. If D is bounded set, the reverse implication is true.*

Example 1.1.27. *In the infinite-dimensional Banach space $(\mathcal{C}([a,b]), \|.\|_\infty)$ consider two integral operators T and S defined by:*

$$\begin{aligned} Ty(t) &= \int_a^b K(t,s,y(s))ds, \\ Sy(t) &= \int_a^t K(t,s,y(s))ds, \;\; t \in [a,b], \end{aligned}$$

where $K : [a,b] \times [a,b] \times [-r,r] \to \mathbb{R}$ is a continuous function. Set

$$M = \{x \in \mathcal{C}([a,b]), \mathbb{R}) : \|x\| \leq r\}.$$

Then, the operators S and T map M into $\mathcal{C}([a,b], \mathbb{R})$ and are completely continuous.

Proof. We will consider the operator S. The remaining case is treated similarly.

The set $A = [a,b] \times [a,b] \times [-r,r]$ is compact, hence K is bounded and uniformly continuous on A. Thus, there is $\delta > 0$ such that $|K(t,s,y)| \leq \delta$, for all $(t,s,x) \in A$, and for every $\varepsilon > 0$ there exists $\rho = \rho(\varepsilon) > 0$ such that

$$|K(t_1,s_1,y_1) - K(t_2,s_2,y_2)| < \varepsilon,$$

for all (t_i,s_i,x_i) in A, $i = 1,2$, satisfying $|t_1 - t_2| + |s_1 - s_2| + |y_1 - y_2| < \rho$.

(i) The operator S is continuous on M. In fact, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in M with

$$\|y_n - y\|_\infty \to 0, \;\; \text{as } n \to \infty.$$

From the uniform continuity of K and the uniform convergence of the functions y_n to y, Lebesgue's dominated convergence theorem leads

$$\begin{aligned}\|Sy_n - Sy\|_\infty &= \max_{a\le t\le b} |Sy_n(t) - Sy(t)| \\ &= \max_{a\le t\le b} |\int_a^t (K(t,s,y_n(s)) - K(t,s,y(s)))\,ds| \\ &\to 0, \ \ as\ n\to\infty.\end{aligned}$$

(ii) The set $S(M)$ is relatively compact. In fact, let $y \in M$ and $\epsilon > 0$, then

$$|Sy(t)| \le |\int_a^t K(t,s,y(s))ds| \le (b-a)\delta, \ \ for\ all\ t\in[a,b].$$

Furthermore, for $|t_1 - t_2| \le \min(\rho,\varepsilon)$, we have

$$\begin{aligned}|Sy(t_1) - Sy(t_2)| &= \left|\int_a^{t_1} K(t_1,s,y(s)ds - \int_a^{t_2} K(t_2,s,y(s))ds\right| \\ &= \left|\int_a^{t_1} K(t_1,s,y(s))ds - \int_a^{t_1} K(t_2,s,y(s))ds\right. \\ &\quad \left. - \int_{t_1}^{t_2} K(t_2,s,y(s))ds\right| \\ &\le (b-a)\varepsilon + |t_1 - t_2|\delta \le ((b-a)+\delta)\varepsilon.\end{aligned}$$

Hence, by the Ascoli-Arzelà Theorem, the set $S(M)$ is relatively compact.

Consequently, (i) and (ii) together imply that the operator S is completely continuous. □

Remark 1.1.28. *If the operators T_1, $T_2 : D \subset E \longrightarrow F$ are bounded (resp. completely continuous) then for every $\alpha, \beta \in \mathbb{R}$, the operator $\alpha T_1 + \beta T_2$ is bounded (resp. completely continuous).*

II. k-set contractions maps

Definition 1.1.29. *Let E and F be Banach spaces. Let $f : E \longrightarrow F$ be a continuous and bounded mapping.*

1. *f is said to be k-set contraction if there exist $k \geq 0$, such that*

$$\alpha\left(f\left(A\right)\right) \leq k\alpha\left(A\right), \quad \text{for every } A \text{ bounded set of } E, \tag{1.1.1}$$

where α is the KMNC in E.

2. *f is called strict set contraction if $0 \leq k < 1$.*

3. *f is said to be condensing if*

$$\alpha\left(f\left(A\right)\right) < \alpha\left(A\right), \text{ for all bounded and not relatively compact set } A \text{ i.e.}, \alpha\left(A\right) > 0,$$

Proposition 1.1.30. *[40, Proposition 2] Let E be a Banach space and G a subset of E.*
(a) If $T_i : G \to E$ is k_i-set contraction, $i = 1, 2$, then $T_1 + T_2 : G \to E$ is $(k_1 + k_2)$-set contraction.
(b) If $T_1 : G \to E$ is k_1-set contraction and $T_3 : T_1(G) \to E$ is k_3-set contraction, then $T_3 \circ T_1 : G \to E$ is k_1k_3-set contraction.
(c) $T : G \to E$ is completely continuous if and only if T is 0-set contraction.
(d) If $T : G \to E$ is L-Lipschitzian (i.e., $\|T(x) - T(y)\| \leq L\|x - y\|$ for $x, y \in G$), then T is k-set contraction with $k = L$.
(f) If $T : G \to E$ is completely continuous and $S : G \to E$ is L-Lipschitzian, then $T + S$ is k-set contraction with $k = L$.
(g) If $T : G \to E$ is completely continuous and $S : G \to E$ is contraction with constant k, then $T + S$ is strict set contraction with constant k.

Proposition 1.1.31. *Let E be a Banach space and D a subset of E. Suppose that $T : D \subset E \longrightarrow E$ is a k-set contraction and $\gamma : D \longrightarrow \mathbb{R}^+$ is continuous function such that $\sup\left\{\gamma\left(x\right) : x \in D\right\} = \ell$. Let $\widetilde{T} : D \subset E \longrightarrow E$ be a map defined by*

$$\widetilde{T}\left(x\right) = \gamma\left(x\right)T\left(x\right), \ \forall x \in D.$$

Then, $\widetilde{T}$ is a $k\ell$-set contraction.

Proof. $\widetilde{T}$ is continuous and bounded as a product of two maps continuous and bounded. Let Ω be a bounded subset of D. We have

$$\widetilde{T}(\Omega) \subset conv\left(\{0\} \cup \ell\, T\,(\Omega)\right).$$

By the properties $(1)\,,(3)\,,(5)$ and (9) of the KMNC $\alpha(.)$ in Proposition 1.1.20, we get

$$\begin{aligned} \alpha(\widetilde{T}(\Omega)) &\leq \alpha\left(conv\left(\{0\} \cup \ell\, T\,(\Omega)\right)\right) \\ &= \alpha\left(\{0\} \cup \ell\, T\,(\Omega)\right) \\ &= \max\left(\alpha(\{0\}), \alpha(\ell\, T\,(\Omega))\right) \\ &= \ell\, \alpha(T\,(\Omega)) \\ &\leq k\ell\alpha(\Omega). \end{aligned}$$

Hence $\widetilde{T}$ is a $k\ell$-set contraction. □

Remark 1.1.32. *Every completely continuous mapping f, is strict set contraction and every strict set contraction is condensing map. Moreover, every condensing map is 1-set contraction.*

Example 1.1.33. *Let X be a real normed space and $T : X \longrightarrow X$ be a linear bounded operator. Then T is a $\|T\|$-set contraction.*

Example 1.1.34. *Let $f \in \mathcal{C}([a,b]$. Define the mapping $T : f \in \mathcal{C}([a,b]) \longrightarrow \mathbb{R}$ by*

$$Tf(t) = \int_a^b cos(f(t))dt,\ f \in \mathcal{C}([a,b]).$$

For any $t \in [a,b]$, *we have*

$$\begin{aligned} |Tf(t) - Tg(t)| &= \left| \int_a^b \cos(f(t)) - \cos(g(t)) dt \right| \\ &\leq \int_a^b |\cos(f(t)) - \cos(g(t))| \, dt \\ &\leq \int_a^b |f(t) - g(t)| \, dt \\ &\leq (b-a) \parallel f - g \parallel_\infty . \end{aligned}$$

So, T *is* $(b-a)$*-set contraction. If* $b-a < 1$, *then* T *is strict set contraction.*

III. Expansive and Lipschitz invertible mappings

Definition 1.1.35. *Let* (X,d) *be a metric space and* D *be a subset of* X. *The mapping* $T : D \to X$ *is said to be expansive if there exists a constant* $h > 1$ *such that*

$$d(Tx, Ty) \geq h \, d(x,y), \quad \forall \, x, y \in D.$$

Example 1.1.36.

(1) An affine function with a leading coefficient $\sigma > 1$ *is* σ*-expansive on* $\mathbb{R}$.

(2) The function $f(x) = x^3 + \theta x$, $x \in \mathbb{R}^+$ *is* θ*-expansive.*

(3) The function $f(x) = \gamma \frac{x}{x+\delta}$, $x \in [a,b]$ *is* $\frac{|\gamma \, \delta|}{(b+\delta)^2}$*-expansive.*

Example 1.1.37. *Let* E *be a infinite dimensional Banach space and let* $T : E \to E$ *be the map defined by:*

$$Tx = \begin{cases} 2x & \text{if} \quad x \in \overline{B(0,1)} \\ \left(1 + \frac{1}{\|x\|}\right) x & \text{if} \quad x \in E \setminus B(0,1). \end{cases}$$

Then T *is 1-expansive. In fact, for* $x, y \in E$, *we distinguish the following four cases:*

Case 1. $x, y \in \overline{B(0,1)}$. *We have* $\|Tx - Ty\| = 2\|x - y\|$.

Case 2. *$x \in \overline{B(0,1)}$ and $y \in E \setminus B(0,1)$. Since $\|x\| \leq 1$, we have $-(\|y\| - \|x\|) \leq -(\|y\| - 1)$. Hence*

$$\begin{aligned}\|Tx - Ty\| &= \left\|2x - y - \frac{y}{\|y\|}\right\| \\ &= \left\|2(x-y) + \frac{(\|y\|-1)y}{\|y\|}\right\| \\ &\geq 2\|x-y\| - (\|y\| - \|x\|) \\ &\geq 2\|x-y\| - \|x-y\| \\ &= \|x-y\|.\end{aligned}$$

Case 3. *$x \in E \setminus B(0,1)$ and $y \in \overline{B(0,1)}$. As in case 2, we obtain $\|Tx - Ty\| \geq \|x - y\|$.*

Case 4. *$x, y \in E \setminus B(0,1)$. Since $\|x\|, \|y\| \geq 1$, we have $-(\|x\| - \|y\|) \leq -(1 - \|y\|)$ and $-(\|y\| - \|x\|) \leq -(1 - \|x\|)$. Hence*

$$\begin{aligned}\|Tx - Ty\| &= \left\|x + \frac{x}{\|x\|} - y - \frac{y}{\|y\|}\right\| \\ &= \left\|2(x-y) + \frac{(1-\|x\|)x}{\|x\|} - \frac{(1-\|y\|)y}{\|y\|}\right\| \\ &\geq \|2(x-y)\| - (1 - \|x\|) - (1 - \|y\|) \\ &\geq 2\|x-y\| - (\|y\| - \|x\|) - (\|x\| - \|y\|) \\ &= 2\|x-y\|.\end{aligned}$$

Let $(X, \|.\|)$ be a linear normed space and $D \subset X$. An operator $A : D \to X$ is said to be γ-Lipschitz invertible on D if it is invertible and its inverse is Lipschitzian on $A(D)$ with constant γ. In what follows we give some examples.

Example 1.1.38.

(1) The function $f(x) = \tan(x)$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is 1-Lipschitz invertible on $\mathbb{R}$.

(2) An affine function with a leading coefficient σ is $\frac{1}{\sigma}$-Lipschitz invertible on $\mathbb{R}$.

Lemma 1.1.39. *[47, Lemma 2.1] Let* $(X, \|.\|)$ *be a linear normed space and* $D \subset X$*. Assume that the mapping* $T : D \to X$ *is expansive with constant* $h > 1$*. Then the inverse of* $I - T : D \to (I - T)(D)$ *exists and*

$$\|(I-T)^{-1}x - (I-T)^{-1}y\| \leq \frac{1}{h-1}\|x-y\|, \quad \forall\, x, y \in (I-T)(D).$$

Lemma 1.1.40. ([48, Lemma 2.3]) *Let* $(E, \|.\|)$ *be a Banach space and* $T : E \to E$ *be Lipschitzian map with constant* $\beta > 0$*. Assume that for each* $z \in E$*, the map* $T_z : E \to E$ *defined by* $T_z x = Tx + z$ *satisfies that* T_z^p *is expansive and onto for some* $p \in \mathbb{N}$*. Then* $(I - T)$ *maps* E *onto* E*, the inverse of* $I - T : E \to E$ *exists, and*

$$\|(I-T)^{-1}x - (I-T)^{-1}y\| \leq \gamma_p\|x-y\| \;\; \textit{for all} \;\;\; x, y \in E,$$

where

$$\gamma_p = \frac{\beta^p - 1}{(\beta - 1)(lip(T^p) - 1)},$$

with $lip(T^p) = \max\{h \geq 0 : \; d(T^p x, T^p y) \geq h\, d(x, y), \; \forall\, x, y \in E\}$*.*

Lemma 1.1.41. ([48, Lemma 2.5]) *Let* $(X, \|.\|)$ *be a linear normed space,* $M \subset X$*. Assume that* $T : M \to X$ *is a contraction with a constant* $k < 1$*, then the inverse of* $I - T : M \to (I - T)(M)$ *exists, and*

$$\|(I-T)^{-1}x - (I-T)^{-1}y\| \leq (1-k)^{-1}\|x-y\| \quad \textit{for all} \;\; x, y \in (I-T)(M).$$

1.2 Fixed point index

The Leray-Schauder degree is an important tool in nonlinear analysis, allowing to establish the existence of fixed points for a mapping acting in a normed linear space. There are many interesting problems not set on the whole space, but instead the setting is a closed convex subset of a normed linear space, e.g., a cone. The fixed point index is a generalization of the Leray-Schauder degree, that is designed to find fixed points of maps defined on a closed convex subset of a Banach space that is not a vector subspace.

Early in the 1970s, Amann [2, 1] and Nussbaum [37, 38] introduced the fixed point index for strict set contractions and condensing mappings and have derived as results some fixed point theorems. As an extension, recently, Djebali and Mebarki [16] have developed a generalized fixed point index theory for the sum of an h-expansive mapping and a k-set contraction when $0 \leq k < h-1$ as well as in the limit case $k = h-1$. Then some researchers have been interested in the extension of this index in various directions, we cite [11, 17, 26].

This section starts with a reminder of the main properties of the fixed point index for strict set contractions in a retract of a Banach space. Then we will present the fixed point index for some classes of sums of two mappings. We will consider separately two cases: firstly the case of the sum $T+F$, where T is an h-expansive map and F is a k-set contraction when $0 \leq k < h-1$ is treated. The definition of a generalized fixed point index as well as some of its properties are presented. Then several results allowing computation of this index are shown. Secondly, we extend some of these results to the case of the sum $T+F$, where T is a mapping such that $(I-T)$ is Lipschitz invertible and F is a k-set contraction.

Definition 1.2.1 (Retracted set)**.** *Let E be a Banach space. A subset X of E is called a retract of E, if there exists a continuous mapping $r : E \to X$ such that*

$$r(x) = x, \ \forall x \in X.$$

Then the mapping r is called retraction.

Remark 1.2.2. *Let $B(0,\rho) = \{x \in E : \|x\| \leq \rho\}$ be the closed ball in E with center 0 and radius ρ. Then $r : E \to B(0,\rho)$ given by*

$$r(y) = \begin{cases} y, & \text{for } \|y\| \leq \rho; \\ \rho y/\|y\|, & \text{for } \|y\| > \rho, \end{cases}$$

defines a retraction (called the standard retraction) of E onto $B(0,\rho)$.

Remark 1.2.3. *Every closed convex set of a Banach space E is a retract of E, in particular every cone $\mathcal{P} \subset E$ is a retract of E.*

1.2.1 Fixed point index for strict set contractions

The development of the theory of the fixed point index for sums of two operators, which will be presented in subsection 1.2.2, involves the fixed point index for strict set contractions whose basic properties are collected in the following theorem.

Theorem 1.2.4. *[28, Theorem 1.3.5]. Let X be a retract of a Banach space E. For every bounded open subset $U \subset X$ and every strict set contraction $f : \overline{U} \to X$ without fixed point on the boundary ∂U, there exists uniquely one integer $i\,(f, U, X)$ satisfying the following conditions:*

(a) (Normalization property). If $f : \overline{U} \to U$ is a constant map, then

$$i\,(f, U, X) = 1.$$

(b) (Additivity property). For any pair of disjoint open subsets U_1, U_2 in U such that f has no fixed point on $\overline{U} \setminus (U_1 \cup U_2)$, we have

$$i\,(f, U, X) = i\,(f, U_1, X) + i\,(f, U_2, X),$$

where $i\,(f, U_j, X) := i\,(f|_{\overline{U_j}}, U_j, X),\ j = 1, 2.$

(c) (Homotopy Invariance property). The index $i\,(h(t, .), U, X)$ does not depend on the

parameter $t \in [0,1]$, *where*

(i) $h : [0,1] \times \overline{U} \to X$ *is continuous and* $h(t,x)$ *is uniformly continuous in* t *with respect to* $x \in \overline{U}$,

(ii) $h(t,.) : \overline{U} \to X$ *is a strict* k*-set contraction, where* k *does not depend on* $t \in [0,1]$,

(iii) $h(t,x) \neq x$, *for every* $t \in [0,1]$ *and* $x \in \partial U$.

(d) (Preservation property). If Y *is a retract of* X *and* $f(\overline{U}) \subset Y$, *then*

$$i\,(f, U, X) = i\,(f, U \cap Y, Y),$$

where $i\,(f, U \cap Y, Y) := i\,(f|_{\overline{U \cap Y}}, U, Y)$.

(e) (Excision property). Let $V \subset U$ *an open subset such that* f *has no fixed point in* $\overline{U} \backslash V$. *Then*

$$i\,(f, U, X) = i\,(f, V, X).$$

(f) (Solvability property). If $i\,(f, U, X) \neq 0$, *then* f *has a fixed point in* U.

The following results are direct consequences of the properties of the index i.

Proposition 1.2.5. *Let* X *be a closed convex of a Banach space* E *and* $U \subset X$ *a bounded open subset with* $0 \in U$. *Assume that* $A : \overline{U} \to X$ *is a strict set contraction that satisfies the Leray-Schauder boundary condition:* $Ax \neq \lambda x$, $\forall\, x \in \partial U$, $\forall\, \lambda \geq 1$. *Then* $i\,(f, U, X) = 1$.

Proof. Consider the homotopic deformation $H : [0,1] \times X \cap \overline{U} \to X$ defined by

$$H(t,x) = tAx.$$

Then the map H is continuous and $H(t,\cdot) : \overline{U} \to X$ is a strict set contraction, and has no fixed point on $\mathcal{P} \cap \partial U$, $\forall\, t \in [0,1]$; otherwise:

- If $t = 0$, there exists some $x_0 \in \partial U$ such that $x_0 = 0$, contradicting $x_0 \in U$.

• If $t \in (0,1]$, there exists some $x_0 \in \partial U$ such that $tAx_0 = x_0$; then $Ax_0 = \frac{1}{t}x_0$ with $\frac{1}{t} \geq 1$, contradicting the assumption. From the invariance under homotopy and the normalization properties of the index, we deduce $i\,(A,U,X) = i\,(0,U,X) = 1$. □

In the sequel, we give an extension of the Leray-Schauder boundary condition, which allows to increase the field of applications of this condition. First, we present our result for the completely continuous mappings.

Proposition 1.2.6. *[13] Let X be a closed convex subset of a Banach space E and $U \subset X$ a bounded open subset with $0 \in U$. Assume $A : \overline{U} \to X$ is a completely continuous mapping without fixed point on the boundary ∂U with $\gamma = dist(0,(I-A)(\partial U))$ and there exists $\varepsilon > 0$ small enough such that*

$$Ax \neq \lambda x \quad \text{for all } x \in \partial U \text{ and } \lambda \geq 1+\varepsilon. \tag{1.2.1}$$

Then the fixed point index $i\,(A,U,X) = 1$.

Proof. Consider the homotopic deformation $H : [0,1] \times \overline{U} \to X$ defined by

$$H(t,x) = \frac{1}{\varepsilon+1}\,tAx.$$

The operator H is completely continuous and has no fixed point on ∂U, $\forall t \in [0,1]$; otherwise, we may distinguish between two cases:

• If $t = 0$, there exists some $x_0 \in \partial U$ such that $x_0 = 0$, contradicting $0 \in U$.

• If $t \in (0,1]$, there exists some $x_0 \in \partial U$ such that $\frac{1}{\varepsilon+1}\,t\,Ax_0 = x_0$; then

$$Ax_0 = \frac{1+\varepsilon}{t}x_0 \text{ with } \frac{1+\varepsilon}{t} \geq 1+\varepsilon,$$

leading to a contradiction with the hypothesis (1.2.1).

From the invariance under homotopy and the normalization properties of the index (see [27, Theorem 2.3.1]), we deduce $i\,(\frac{1}{\varepsilon+1}\,A,U,X) = i\,(0,U,X) = 1$.

Now, we show that $i\,(A,U,X) = i\,(\frac{1}{\varepsilon+1}\,A,U,X)$.

Since A has no fixed point in ∂U and $(I-A)(\partial U)$ is a closed set (see [41, Lemma 1]), we get $0 \notin \overline{(I-A)(\partial U)}$. Hence, $\inf\limits_{x\in\partial U} \|x - Ax\| := \gamma > 0$.

Let ε be sufficiently small so that $\|\frac{\varepsilon}{\varepsilon+1}\,Ax\| < \frac{\gamma}{2}$. Hence

$$\|Ax - \frac{1}{\varepsilon+1}\,Ax\| = \|Ax - Ax + \frac{\varepsilon}{\varepsilon+1}\,Ax\| = \|\frac{\varepsilon}{\varepsilon+1}\,Ax\| < \frac{\gamma}{2},\ \forall\, x \in \partial U.$$

Define the convex deformation $G : [0,1] \times \overline{U} \to X$ by

$$G(t,x) = tAx + (1-t)\frac{1}{\varepsilon+1}Ax.$$

The operator G is completely continuous and has no fixed point on ∂U, $\forall t \in [0,1]$. In fact, for all $x \in \partial U$ and $t \in [0,1]$, we have

$$\begin{aligned}
\|x - G(t,x)\| &= \|x - tAx - (1-t)\tfrac{1}{\varepsilon+1}Ax\| \\
&\geq \|x - \tfrac{1}{\varepsilon+1}Ax\| - t\|Ax - \tfrac{1}{\varepsilon+1}Ax\| \\
&\geq \|x - Ax\| - \|\tfrac{\varepsilon}{\varepsilon+1}Kx\| - t\|Kx - \tfrac{1}{\varepsilon+1}Ax\| \\
&> \gamma - \tfrac{\gamma}{2} - \tfrac{\gamma}{2} = 0.
\end{aligned}$$

Then our claim follows from the homotopy invariance property of the index. □

Remark 1.2.7. *The result of Proposition 1.2.6 remains true if A is a strict contraction and even if A is condensing map and that according to [41, Lemma 1].*

Proposition 1.2.8. *[28, Theorem 1.3.8] Let X be a closed convex of a Banach space E and $U \subset X$ be a bounded open subset. Assume that $A : \overline{U} \to X$ is a strict set contraction. If there exists $u_0 \in X, u_0 \neq 0$, such that $\lambda u_0 \in X$, $\forall\, \lambda \geq 0$ and*

$$x - Ax \neq \lambda u_0,\ \forall\, x \in \partial U,\ \forall\, \lambda \geq 0,$$

then the fixed point index $i\,(A,U,X) = 0$.

Proof. Define the homotopy $H : [0,1] \times \overline{U} \to X$ by $H(t,x) = Ax + t\lambda_0 u_0$, for some

$$\lambda_0 > \sup_{x \in \overline{U}} (\|u_0\|^{-1}(\|x\| + \|Ax\|)). \tag{1.2.2}$$

Such a choice is possible since U is a bounded subset and so is $A(\overline{U})$. The operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t,.)$ is a strict set contraction for each $t \in [0,1]$. In addition, $H(t,.)$ has no fixed point on ∂U. On the contrary, there would exist some $x_0 \in \partial U$ and $t_0 \in [0,1]$ such that

$$x_0 = Ax_0 + t_0 \lambda_0 u_0,$$

contradicting the hypothesis. By Theorem 1.2.4, we get

$$i(A,U,X) = i(H(0,.),U,\mathcal{K}) = i(H(1,.),U,X) = 0. \tag{1.2.3}$$

Indeed, suppose that $i(H(1,.),U,X) \neq 0$. Then there exists $x_0 \in U$ such that $Ax_0 + \lambda_0 u_0 = x_0$, which implies that $\lambda_0 \leq \|u_0\|^{-1}(\|x_0\| + \|Ax_0\|)$, contradicting (1.2.2). □

Proposition 1.2.9. *Assume that X is a closed convex set of a Banach space E, X_1 is a bounded closed convex subset of X, U is a nonempty open set of X with $U \subset X_1$. If $A : X_1 \to X$ is a strict set contraction, $A(X_1) \subset X_1$ and A has no fixed point in $X_1 \backslash U$, then $i(A,U,X) = 1$.*

Proof. Since X_1 is a closed subset of E, $\overline{U} \subset X_1$, by the preservation property of the fixed point index, it follows that

$$i(A,U,X) = i(A,U,X_1). \tag{1.2.4}$$

Because A has no fixed points in $X_1 \backslash U$, by the excision property of the fixed point index, we get

$$i(A,U,X_1) = i(A,X_1,X_1). \tag{1.2.5}$$

Take $z_0 \in U \subset X_1$ and let

$$H(t, x) = tz_0 + (1 - t)Ax, \quad t \in [0, 1], \quad x \in X_1.$$

We have $H : [0, 1] \times X_1 \to X_1$ is continuous and bounded. Also, for any $t \in [0, 1]$ and B a bounded set in X_1, we have

$$\alpha(H(t, B)) \leq (1 - t)\alpha(A(B)) \leq (1 - t)k\alpha(B).$$

So, $H(t, \cdot) : X_1 \to X_1$ is a strict set contraction for any $t \in [0, 1]$. Hence, using the normality and the homotopy invariance of the fixed point index, we get

$$\begin{aligned} i(A, X_1, X_1) &= i(H(0, \cdot), X_1, X_1) \\ &= i(H(1, \cdot), X_1, X_1) \\ &= i(z_0, X_1, X_1) \\ &= 1. \end{aligned}$$

From here and from (1.2.4), (1.2.5), we arrive at

$$i(A, U, X) = i(A, U, X_1) = i(A, X_1, X_1) = 1.$$

□

Corollary 1.2.10. *Assume that X is a closed convex set in E and U is a nonempty bounded open convex subset of X. If $A : \overline{U} \to X$ is a strict set contraction and $A(\overline{U}) \subset U$, then*

$$i(A, U, X) = 1.$$

Proof. We apply Theorem 1.2.9 for $X_1 = \overline{U}$. Then

$$i(A, U, X) = 1.$$

□

1.2.2 Fixed point index for perturbed k-set contractions maps

In all what follows, $\mathcal{P}$ will refer to a cone in a Banach space E, Ω is a subset of $\mathcal{P}$, and U is a bounded open subset of $\mathcal{P}$. For some constant $r > 0$, we will denote $\mathcal{P}_r = \mathcal{P} \cap \mathcal{B}_r$, where $\mathcal{B}_r = \{x \in E : \|x\| < r\}$ is the open ball centered at the origin with radius r.

1. Case of k-set contraction perturbed by an h-expansive map

Assume that $T : \Omega \to E$ is an h-expansive mapping and $F : \overline{U} \to E$ is a k-set contraction. By Lemma 1.1.39, the operator $(I-T)^{-1}$ is $(h-1)^{-1}$-Lipschitzian on $(I-T)(\Omega)$. Suppose that

$$F(\overline{U}) \subset (I-T)(\Omega) \tag{1.2.6}$$

and

$$x \neq Tx + Fx, \quad \text{for all } \; x \in \partial U \cap \Omega. \tag{1.2.7}$$

Then $x \neq (I-T)^{-1}Fx$, for all $x \in \partial U$ and the mapping $(I-T)^{-1}F : \overline{U} \to \mathcal{P}$ is strict $k(h-1)^{-1}$-set contraction. Indeed, $(I-T)^{-1}F$ is continuous and bounded; and for any bounded set B in U, we have

$$\alpha(((I-T)^{-1}F)(B)) \leq (h-1)^{-1}\,\alpha(F(B)) \leq k(h-1)^{-1}\alpha(B).$$

By Lemma 1.2.4, the fixed point index $i\left((I-T)^{-1}F, U, \mathcal{P}\right)$ is well defined. Thus we put

$$i_*\left(T+F, U\cap\Omega, \mathcal{P}\right) = \begin{cases} i\left((I-T)^{-1}F, U, \mathcal{P}\right) & \text{if } U\cap\Omega \neq \emptyset \\ 0, & \text{if } U\cap\Omega = \emptyset. \end{cases} \tag{1.2.8}$$

This integer is called the generalized fixed point index of the sum $T+F$ on $U\cap\Omega$ with respect to the cone $\mathcal{P}$.

Using the main properties of the fixed point index for strict set contractions (in particular, completely continues maps), Djebali and Mebarki, have discussed the properties of the generalized fixed point index i_* in [16].

Theorem 1.2.11. *[16, Theorem 2.3] The fixed point index defined in (1.2.8) satisfies the following properties:*

(a) (Normalization property). If $U = \mathcal{P}_r,\ 0 \in \Omega,$ *and* $Fx = z_0 \in \mathcal{B}(-T0, (h-1)r) \cap \mathcal{P}$ *for all* $x \in \overline{\mathcal{P}_r}$*, then* $i_*\,(T + F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

(b) (Additivity property). For any pair of disjoint open subsets U_1, U_2 *in* U *such that* $T+F$ *has no fixed point on* $(\overline{U} \setminus (U_1 \cup U_2)) \cap \Omega$*, we have*

$$i_*\,(T + F, U \cap \Omega, \mathcal{P}) = i_*\,(T + F, U_1 \cap \Omega, \mathcal{P}) + i_*\,(T + F, U_2 \cap \Omega, \mathcal{P}),$$

where $i_*\,(T + F, U_j \cap \Omega, X) := i_*\,(T + F|_{\overline{U_j}}, U_j \cap \Omega, \mathcal{P}), \quad j = 1, 2.$

(c) (Homotopy Invariance property). The fixed point index $i_*\,(T + H(t, .), U \cap \Omega, \mathcal{P})$ *does not depend on the parameter* $t \in [0, 1]$ *whenever*

(i) $H : [0, 1] \times \overline{U} \to E$ *is continuous and* $H(t, x)$ *is uniformly continuous in* t *with respect to* $x \in \overline{U}$,

(ii) $H([0, 1] \times \overline{U}) \subset (I - T)(\Omega)$,

(iii) $H(t, .) : \overline{U} \to E$ *is a* l*-set contraction with* $0 \leq l < h - 1$ *and* l *does not depend on* $t \in [0, 1]$,

(iv) $Tx + H(t, x) \neq x$*, for all* $t \in [0, 1]$ *and* $x \in \partial U \cap \Omega$.

(d) (Solvability property). If $i_*\,(T + F, U \cap \Omega, \mathcal{P}) \neq 0$*, then* $T + F$ *has a fixed point in* $U \cap \Omega$.

Next, we compute the index i_* under certain considerations.

Proposition 1.2.12. *Let U be a bounded open subset of $\mathcal{P}$ with $0 \in U$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < h - 1$, and $F(\overline{U}) \subset (I - T)(\Omega)$. If*

$$Fx \neq (I - T)(\lambda x), \quad \text{for all } x \in \partial U \cap \Omega \;\text{ and }\; \lambda \geq 1,$$

then the fixed point index $i_*(T + F, U \cap \Omega, \mathcal{P}) = 1$.

Proof. The mapping $(I - T)^{-1}F : \overline{U} \to \mathcal{P}$ is a strict $\frac{k}{h-1}$-set contraction and it is readily seen that the following condition of Leray-Schauder type is satisfied

$$(I - T)^{-1}Fx \neq \lambda x, \quad \text{for all } x \in \partial U \;\text{ and }\; \lambda \geq 1.$$

In fact, if there exist $x_0 \in \partial U$ and $\lambda_0 \geq 1$ such that $(I - T)^{-1}Fx_0 = \lambda_0 x_0$. Then $Fx_0 = (I - T)(\lambda_0 x_0)$, which contradicts our assumption. Our claim then follows from (1.2.8) and Proposition 1.2.5. □

Now, we extend the result of Proposition 1.2.6 to the case of the sum $T + F$, where T is an expansive mapping and F is a completely continuous one.

Proposition 1.2.13. *Assume that $T : \Omega \to E$ is an expansive mapping with constant $h > 1$, $F : \overline{U} \to E$ is a completely continuous mapping and $F(\overline{U}) \subset (I - T)(\Omega)$. Suppose that $T + F$ has no fixed point on $\partial U \cap \Omega$. Then we have the following results: If $0 \in U$ and there exists $\varepsilon > 0$ small enough such that*

$$Fx \neq (I - T)(\lambda x) \;\text{ for all }\; \lambda \geq 1 + \varepsilon,\; x \in \partial U \;\text{ and }\; \lambda x \in \Omega,$$

then the fixed point index $i_*(T + F, U \cap \Omega, \mathcal{P}) = 1$.

Proof. The mapping $(I - T)^{-1}F : \overline{U} \to \mathcal{P}$ is completely continuous without fixed point on ∂U and it is readily seen that the following condition is satisfied

$$(I - T)^{-1}Fx \neq \lambda x \quad \text{for all} \quad x \in \partial U \quad \text{and} \quad \lambda \geq 1 + \varepsilon.$$

Then, our claim follows from the definition of i_* and the Proposition 1.2.6.

□

Remark 1.2.14. *Proposition 1.2.13 is an extension of Proposition 1.2.12 in the case where the map F is completely continuous.*

Proposition 1.2.15. *Let U be a bounded open subset of $\mathcal{P}$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < h - 1$, and $F(\overline{U}) \subset (I - T)(\Omega)$. If there exists $u_0 \in \mathcal{P}^*$ such that*

$$Fx \neq (I - T)(x - \lambda u_0), \ \text{for all } \lambda \geq 0 \text{ and } x \in \partial U \cap (\Omega + \lambda u_0), \tag{1.2.9}$$

then the fixed point index $i_* (T + F, U \cap \Omega, \mathcal{P}) = 0$.

Proof. The mapping $(I - T)^{-1}F : \overline{U} \to \mathcal{P}$ is strict $\frac{k}{h-1}$-set contraction and for some $u_0 \in \mathcal{P}^*$ this operator satisfies

$$x - (I - T)^{-1}Fx \neq \lambda u_0, \ \forall\, x \in \partial U, \ \forall\, \lambda \geq 0.$$

By (1.2.8) and Proposition 1.2.8, we deduce that

$$i_* (T + F, U \cap \Omega, \mathcal{P}) = i\left((I - T)^{-1}F, U, \mathcal{P}\right) = 0.$$

□

2. Case of k-set contraction perturbed by a map T where $(I - T)$ is Lipschitz invertible

Let E be a real Banach space and $\mathcal{P}$ be a cone of E, $\Omega \subset \mathcal{P}$ and U is a bounded open subset of $\mathcal{P}$. Assume that $T : \Omega \to E$ be such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction mapping with $0 \leq k < \gamma^{-1}$. Suppose that

$$F(\overline{U}) \subset (I - T)(\Omega),$$

and

$$x \neq Tx + Fx, \text{ for all } x \in \partial U \bigcap \Omega.$$

Then $x \neq (I - T)^{-1}Fx$, for all $x \in \partial U$ and the mapping $(I - T)^{-1}F : \overline{U} \to \mathcal{P}$ is a strict set contraction with constant $k\gamma < 1$. Indeed, $(I - T)^{-1}F$ is continuous and bounded; and for any bounded set B in U, we have

$$\alpha(((I - T)^{-1}F)(B)) \leq \gamma\, \alpha(F(B)) \leq k\gamma\, \alpha(B).$$

The fixed point index $i\,((I - T)^{-1}F, U, \mathcal{P})$ is well defined. Thus we put,

$$i_*\,(T + F, U \bigcap \Omega, \mathcal{P}) = \begin{cases} i\,((I - T)^{-1}F, U, \mathcal{P}), & \text{if } U \bigcap \Omega \neq \emptyset \\ 0, & \text{if } U \bigcap \Omega = \emptyset. \end{cases} \tag{1.2.10}$$

The proof of our theoretical result presented in Chapter 4 invokes the following main properties of the fixed point index i_*.

(i) *(Normalization)* If $Fx = y_0$, for all $x \in \overline{U}$, where $(I - T)^{-1}y_0 \in U \cap \Omega$, then

$$i_*\,(T + F, U \cap \Omega, \mathcal{P}) = 1.$$

(ii) *(Additivity)* For any pair of disjoint open subsets $U_1, U_2 \subset U$ such that $T + F$ has no fixed point on $(\overline{U} \setminus (U_1 \cup U_2)) \cap \Omega$, we have

$$i_*(T + F, U \cap \Omega, \mathcal{P}) = i_*(T + F, U_1 \cap \Omega, \mathcal{P}) + i_*(T + F, U_2 \cap \Omega, \mathcal{P}).$$

(iii) *(Homotopy invariance)* The fixed point index $i_*(T + H(.,t), U \cap \Omega, \mathcal{P})$ does not depend on the parameter $t \in [0, 1]$, where

(a) $H : [0,1] \times \overline{U} \to E$ is continuous and $H(t,x)$ is uniformly continuous in t with respect to $x \in \overline{U}$,

(b) $H([0,1] \times \overline{U}) \subset (I-T)(\Omega)$,

(c) $H(t,.) : \overline{U} \to E$ is a ℓ-set contraction with $0 \leq \ell < \gamma^{-1}$ for all $t \in [0,1]$,

(d) $Tx + H(t,x) \neq x$ for all $t \in [0,1]$ and $x \in \partial U \cap \Omega$.

(vi) *(Solvability)* If $i_*(T+F, U\cap\Omega, \mathcal{P}) \neq 0$, then $T+F$ has a fixed point in $U \cap \Omega$.

For more details about the definition of the index i_* and its properties see [16, 26].

Now, we compute the index i_* under certain considerations.

Proposition 1.2.16. *Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $tF(\overline{U}) \subset (I-T)(\Omega)$ for all $t \in [0,1]$. If $(I-T)^{-1}0 \in U$, and*

$$(I-T)x \neq \lambda Fx \quad \text{for all } x \in \partial U \bigcap \Omega \quad \text{and } 0 \leq \lambda \leq 1, \tag{1.2.11}$$

then the fixed point index $i_*(T+F, U\bigcap\Omega, \mathcal{P}) = 1$.

Proof. Consider the homotopic deformation $H : [0,1] \times \overline{U} \to \mathcal{P}$ defined by

$$H(t,x) = (I-T)^{-1} tFx.$$

The operator H is continuous and uniformly continuous in t for each x. Moreover, $H(t,.)$ is a strict $k\gamma$-set contraction for each t and the mapping $H(t,.)$ has no fixed point on ∂U. Otherwise, there would exist some $x_0 \in \partial U \bigcap \Omega$ and $t_0 \in [0,1]$ such that

$$x_0 - Tx_0 = t_0 F x_0,$$

which contradicts our assumption.

From the invariance under homotopy and the normalization property of the index fixed point, we deduce that

$$i_*\left((I-T)^{-1}F, U, \mathcal{P}\right) = i_*\left((I-T)^{-1}0, U, \mathcal{P}\right) = 1.$$

Consequently, from (1.2.10), we deduce that

$$i_*\left(T+F, U\bigcap\Omega, \mathcal{P}\right) = 1,$$

which completes the proof. □

Proposition 1.2.17. *Let U be a bounded open subset of $\mathcal{P}$ with $0 \in U$. Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{U}) \subset (I-T)(\Omega)$. If*

$$Fx \neq (I-T)(\lambda x) \;\; \textit{for all } x \in \partial U,\ \lambda \geq 1 \;\; \textit{and} \;\; \lambda x \in \Omega,$$

then the fixed point index $i_*\left(T+F, U\bigcap\Omega, \mathcal{P}\right) = 1$.

Proof. The mapping $(I-T)^{-1}F : \overline{U} \to \mathcal{P}$ is strict γk-set contraction and it is readily seen that the following condition of Leray-Schauder type is satisfied

$$(I-T)^{-1}Fx \neq \lambda x, \quad \text{for all } x \in \partial U \;\; \text{and } \lambda \geq 1.$$

In fact, if there exist $x_0 \in \partial U$ and $\lambda_0 \geq 1$ such that $(I-T)^{-1}Fx_0 = \lambda_0 x_0$.

Then $Fx_0 = (I-T)(\lambda_0 x_0)$, which contradicts our assumption. The claim then follows from (1.2.10) and the Proposition 1.2.5. □

Proposition 1.2.18. *Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $F(\overline{U}) \subset (I-T)(\Omega)$. If there exists $u_0 \in \mathcal{P}^*$ such that*

$$Fx \neq (I-T)(x - \lambda u_0), \;\; \textit{for all } \lambda \geq 0 \textit{ and } x \in \partial U\bigcap(\Omega + \lambda u_0), \tag{1.2.12}$$

then the fixed point index $i_*\left(T+F, U\bigcap\Omega, \mathcal{P}\right)=0$.

Proof. The mapping $(I-T)^{-1}F:\overline{U}\to\mathcal{P}$ is strict γk-set contraction and for some $u_0\in\mathcal{P}^*$ this operator satisfies

$$x-(I-T)^{-1}Fx\neq\lambda u_0,\ \forall\, x\in\partial U,\ \forall\,\lambda\geq 0.$$

By (1.2.10) and the Proposition 1.2.8, we deduce that

$$i_*\left(T+F, U\bigcap\Omega, \mathcal{P}\right)=i\left((I-T)^{-1}F, U, \mathcal{P}\right)=0.$$

□

1.3 Some fixed point theorems

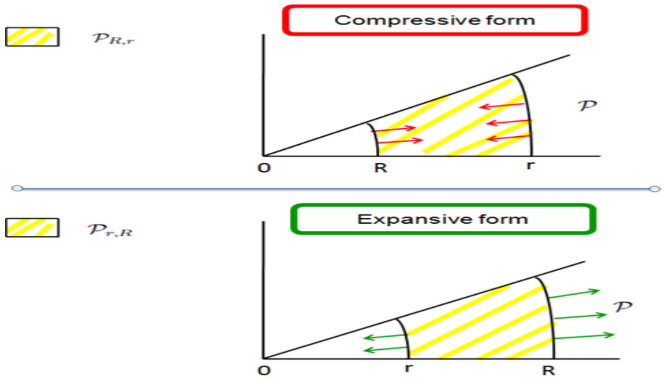
1.3.1 Expansion-compression fixed point theorem of Leggett-Williams type

In what follows, $\mathcal{P}$ will refer to a cone in a Banach space $(E,\|.\|)$. Let χ and ψ be nonnegative continuous functionals on $\mathcal{P}$. For positive real numbers a and b, we define the sets:

$$\mathcal{P}(\chi,b)=\{x\in\mathcal{P}:\chi(x)\leq b\},$$

$$\mathcal{P}(\chi,\psi,a,b)=\{x\in\mathcal{P}:a\leq\chi(x)\text{ and }\psi(x)\leq b\}.$$

The expansion-compression fixed point theorems of Krasnosel'skii type give us fixed points localized in a conical shell of the form $\mathcal{P}_{ab}=\{x\in\mathcal{P}:a\leq\|x\|\leq b\}$ (see [27, 32, 33]) , while with the Leggett-Williams theorems type, the fixed points are localized in a conical shell of the form $\mathcal{P}(\chi,\psi,a,b)$ (see [4, 34]). In [3, Theorem 4.1], Anderson *et al*, have developed a functional expansion-compression fixed point theorem of Leggett-Williams type. They have discussed the existence of at least one solution in $\mathcal{P}(\beta, \alpha, r, R)$ or in $\mathcal{P}(\alpha, \beta, r, R)$ for the nonlinear operational equation $Ax = x$, where A is a completely continuous nonlinear map acting in $\mathcal{P}$, α is a nonnegative continuous concave functional on $\mathcal{P}$ and β is a nonnegative continuous convex functional on $\mathcal{P}$. Noting that, in [3], the authors provided more general results than those obtained in [4, 6, 29, 30, 34, 42] for completely continuous mappings. An illustration of the compressive and the expansive form, for $E = \mathbb{R}^2$, $\chi(x) = \psi(x) = \|x\|$, $x \in \mathbb{R}^2_+$, is depicted in the following figure



Theorem 1.3.1. *[3, Theorem 4.1] Let $\mathcal{P}$ be a cone in a real Banach space E, α is a nonnegative continuous concave functional on $\mathcal{P}$, β is a nonnegative continuous convex functional on $\mathcal{P}$ and $T : \mathcal{P} \longrightarrow \mathcal{P}$ is a completely continuous operator.*

If there exist nonnegative numbers a, b, c and d such that

(A1) $\{x \in \mathcal{P} : \ a < \alpha(x) \ \text{ and } \beta(x) < b\} \neq \emptyset$;

(A2) *if $x \in \mathcal{P}$ with $\beta(x) = b$ and $\alpha(x) \geq a$, then $\beta(Tx) < b$;*

(A3) *if $x \in \mathcal{P}$ with $\beta(x) = b$ and $\alpha(Tx) < a$, then $\beta(Tx) < b$;*

(A4) $\{x \in \mathcal{P} : \ c < \alpha(x) \ \text{ and } \beta(x) < d\} \neq \emptyset$;

(A5) *if $x \in \mathcal{P}$ with $\alpha(x) = c$ and $\beta(x) \leq d$, then $\alpha(Tx) > c$;*

(A6) *if* $x \in \mathcal{P}$ *with* $\alpha(x) = c$ *and* $\beta(Tx) > d$, *then* $\alpha(Tx) > c$;

and if

(H1) $a < c$, $b < d$, $\{x \in \mathcal{P} : \ b < \beta(x)$ *and* $\alpha(x) < c\} \neq \emptyset$, $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$ *and* $\mathcal{P}(\alpha, c)$ *is bounded then* T *has a fixed point* x^* *in* $\mathcal{P}(\beta, \alpha, b, c)$;

(H2) $c < a$, $d < b$, $\{x \in \mathcal{P} : \ a < \alpha(x)$ *and* $\beta(x) < d\} \neq \emptyset$, $\mathcal{P}(\alpha, a) \subset \mathcal{P}(\beta, d)$ *and* $\mathcal{P}(\beta, d)$ *is bounded then* T *has a fixed point* x^* *in* $\mathcal{P}(\alpha, \beta, a, d)$.

Proof. We will prove the expansion result $(H1)$. The proof of the compression result $(H2)$ is similar. Let

$$U = \{x \in \mathcal{P} : \beta(x) < b\},$$

$$V = \{x \in \mathcal{P} : \alpha(x) < c\},$$

Then, the interior of $V - U$ is given by

$$W = (V - U)^o = \{x \in V : \ b < \beta(x) \text{ and } \alpha(x) < c\},$$

Thus U, V and W are bounded (they are subsets of V which is bounded by condition $(H1)$), non-empty (by conditions (A1), (A4) and (H1)) and open subsets of $\mathcal{P}$. To prove the existence of a fixed point for our operator T in $\mathcal{P}(\beta, \alpha, b, c)$, it is enough for us to show that $i(T, W, \mathcal{P}) \neq 0$ since W is the interior of $\mathcal{P}(\beta, \alpha, b, c)$.

Claim 1. $Tx \neq x$ for all $x \in \partial U$. In fact,

Let $z_0 \in \partial U$, then $\beta(z_0) = b$. Suppose that $z_0 = Tz_0$. If $\alpha(Tz_0) < a$ then $\beta(Tz_0) < b$ by condition (A3), and if $\alpha(z_0) = \alpha(Tz_0) \geq a$ then $\beta(Tz_0) < b$ by condition (A2). Hence in either case we have that $z_0 \neq Tz_0$, thus T does not have any fixed points on ∂U.

Claim 2. $Tx \neq x$ for all $x \in \partial V$. In fact,

Let $z_1 \in \partial V$, then $\alpha(z_1) = c$. Suppose that $z_1 = Tz_1$. If $\beta(Tz_1) > d$ then $\alpha(Tz_1) > c$ by condition (A6), and if $\beta(z_1) = \beta(Tz_1) \leq d$ then $\alpha(Tz_1) > c$ by condition (A5). Hence in either case we have that $z_1 \neq Tz_1$, thus T does not have any fixed points on ∂V.

Let $\omega_1 \in \{x \in \mathcal{P} : \ a < \alpha(x) \text{ and } \beta(x) < b\}$ and let $H_1 : [0,1] \times \overline{U} \to \mathcal{P}$ be defined by

$$H_1(t,x) = (1-t)Tx + t\omega_1.$$

Clearly, H_1 is continuous and $H_1([0,1] \times \overline{U})$ is relatively compact.

Claim 3. $H_1(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial U$. In fact,

Suppose the contrary, that is there exist $(t_1,x_1) \in [0,1] \times \partial U$ such that $H_1(t_1,x_1) = x_1$. Since $x_1 \in \partial U$ we have that $\beta(x_1) = b$. Either $\alpha(Tx_1) < a$ or $\alpha(Tx_1) \geq a$.

Case (1): $\alpha(Tx_1) < a$. By condition (A3) we have $\beta(Tx_1) < b$, which is a contradiction since

$$\begin{aligned} b = \beta(x_1) &= \beta\left((1-t_1)Tx_1 + t_1\omega_1\right) \\ &\leq (1-t_1)\beta(Tx_1) + t_1\beta(\omega_1) \\ &< b. \end{aligned}$$

Case (2): $\alpha(Tx_1) \geq a$. We have that $\alpha(x_1) \geq a$ since

$$\begin{aligned} \alpha(x_1) &= \alpha\left((1-t_1)Tx_1 + t_1\omega_1\right) \\ &\geq (1-t_1)\alpha(Tx_1) + t_1\alpha(\omega_1) \\ &\geq a, \end{aligned}$$

and thus by condition (A2) we have $\beta(Tx_1) < b$, which is the same contradiction we arrived at in the previous case.

Therefore, we have shown that $H_1(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial U$, and thus by the homotopy invariance property of the index i

$$i(T,U,\mathcal{P}) = i(\omega_1,U,\mathcal{P})$$

and by the normality property of the index i

$$i(T,U,\mathcal{P}) = i(\omega_1,U,\mathcal{P}) = 1.$$

Let $\omega_2 \in \{x \in \mathcal{P} : \ c < \alpha(x) \text{ and } \beta(x) < d\}$ and let $H_2 : [0,1] \times V \to \mathcal{P}$ be defined by:

$$H_1(t,x) = (1-t)Tx + t\omega_2.$$

Clearly, H_2 is continuous and $H_2([0,1] \times \overline{V})$ is relatively compact.

Claim 4. $H_2(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial V$. In fact,

Suppose not; that is, there exist $(t_2, x_2) \in [0,1] \times \partial V$ such that $H_2(t_2, x_2) = x_2$.

Since $x_2 \in \partial V$ we have that $\alpha(x_2) = c$. Either $\beta(Tx_2) \leq d$ or $\beta(Tx_2 > d$.

Case (1): $\beta(Tx_2 > d$. By condition (A6) we have $\alpha(Tx_2) > c$, which is a contradiction since

$$\begin{aligned} c = \alpha(x_2) &= \alpha\left((1-t_2)Tx_2 + t_2\omega_2\right) \\ &\geq (1-t_2)\alpha(Tx_2) + t_2\alpha(\omega_2) \\ &> c. \end{aligned}$$

Case (2): $\beta(Tx_2) \leq d$. We have that $\beta(x_2) \leq d$ since

$$\begin{aligned} \beta(x_1) &= \beta\left((1-t_2)Tx_2 + t_2 w_2\right) \\ &\leq (1-t_2)\beta(Tx_2) + t_2\beta(w_2) \\ &\leq d, \end{aligned}$$

and thus by condition(A5) we have $\alpha(Tx_2) > c$, which is the same contradiction we arrived at in the previous case.

Therefore, we have shown that $H_2(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial V$ and thus by the homotopy invariance property of the index i

$$i(T, V, \mathcal{P}) = i(\omega_2, V, \mathcal{P}).$$

By the solution property of the index i (since $\omega_2 \notin V$ the index cannot be nonzero) we have

$$i(T, V, \mathcal{P}) = i(\omega_2, V, \mathcal{P}) = 0.$$

Since U and W are disjoint open subsets of V and T has no fixed points in $\overline{V} - (U \cup V)$ (by claims 1 and 2), by the additivity property of the index i

$$i(T, V, \mathcal{P}) = i(T, U, \mathcal{P}) + i(T, W, \mathcal{P}).$$

Consequently, we have

$$i(T, W, \mathcal{P}) = -1$$

and thus by the solution property of the index i the operator T has a fixed point

$$x^* \in W \subset \mathcal{P}\,(\beta, \alpha, b, c)\,.$$

□

1.3.2 Fixed point theorem for the sum of two operators

To prove one of our existence results, in chapter 3, we will use Theorem 1.3.2, that we will present and demonstrate in the sequel.

Theorem 1.3.2. *[25] Let $\epsilon > 0$, $R > 0$, E be a Banach space and $X = \{x \in E : \|x\| \leq R\}$. Let also, $Tx = -\epsilon x$, $x \in X$, $S : X \to E$ is a continuous map such that $(I - S)(X)$ resides in a compact subset of E and*

$$\{x \in E : x = -\lambda(I - S)x, \quad \|x\| = R\} = \emptyset \tag{1.3.1}$$

for any $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then there exists $x^* \in X$ such that $Tx^* + Sx^* = x^*$.

Proof. Define

$$r\left(\frac{1}{\epsilon}x\right) = \begin{cases} \frac{1}{\epsilon}x & \text{if} \quad \|x\| \leq \epsilon R \\ \frac{Rx}{\|x\|} & \text{if} \quad \|x\| > \epsilon R. \end{cases}$$

Then $r\left(-\frac{1}{\epsilon}(I-S)\right) : X \to X$ is continuous and compact.

Suppose that $r\left(-\frac{1}{\epsilon}(I-S)\right) \neq x$ for $x \in \partial X$, otherwise we are finished. From Proposition 1.2.9 and the existence property of the fixed point index, it follows that there exists $x^* \in \mathring{X}$ so that

$$r\left(-\frac{1}{\epsilon}(I-S)x^*\right) = x^*.$$

Assume that $-\frac{1}{\epsilon}(I-S)x^* \notin X$. Then

$$\left\|(I-S)x^*\right\| > R\epsilon, \quad \frac{R}{\|(I-S)x^*\|} < \frac{1}{\epsilon}$$

and

$$x^* = -\frac{R}{\|(I-S)x^*\|}(I-S)x^* = r\left(-\frac{1}{\epsilon}(I-S)x^*\right)$$

and hence, $\|x^*\| = R$. This contradicts with (1.3.1). Therefore $-\frac{1}{\epsilon}(I-S)x^* \in X$ and

$$x^* = r\left(-\frac{1}{\epsilon}(I-S)x^*\right) = -\frac{1}{\epsilon}(I-S)x^*$$

or

$$-\epsilon x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*.$$

This completes the proof. □

1.3.3 Multiple fixed point theorem for the sum of two operators

In this section, we present a multiple fixed point theorem. The proof rely on the results of Propositions 1.2.13 and 1.2.15 producing the computation of the index i_*. The following result will be used to prove the existence of at least two nonnegative solutions to the problem (3.1.1)-(3.1.2).

Theorem 1.3.3. *Let $\mathcal{P}$ be a cone in a Banach space $(E, \|.\|)$. Let Ω be a subset of $\mathcal{P}$, and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to E$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a completely continuous mapping and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) $Sx \neq (I - T)(x - \lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,

(ii) *there exists $\varepsilon > 0$ small enough such that* $Sx \neq (I - T)(\lambda x)$, *for all* $\lambda \geq 1 + \varepsilon$, $x \in \partial U_2$ *and* $\lambda x \in \Omega$,

(iii) $Sx \neq (I - T)(x - \lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Proof. If $Sx = (I - T)x$ for $x \in \partial U_2 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_2 \cap \Omega$ of the operator $T + S$. Suppose that $Sx \neq (I - T)x$ for any $x \in \partial U_2 \cap \Omega$. Without loss of generality, assume that $Tx + Sx \neq x$ on $\partial U_1 \cap \Omega$ and $Tx + Sx \neq x$ on $\partial U_3 \cap \Omega$, otherwise the result is obvious. By Propositions 1.2.13 and 1.2.15, we have

$$i_* (T + S, U_1 \cap \Omega, \mathcal{P}) = i_* (T + S, U_3 \cap \Omega, \mathcal{P}) = 0,$$

and

$$i_* (T + S, U_2 \cap \Omega, \mathcal{P}) = 1.$$

The additivity property of the index i_* yields

$$i_*\left(T+S,(U_2\setminus \overline{U}_1)\cap\Omega,\mathcal{P}\right)=1\ \text{ and }\ \ i_*\left(T+S,(U_3\setminus \overline{U}_2)\cap\Omega,\mathcal{P}\right)=-1.$$

Consequently, by the existence property of the index i_*, $T+S$ has at least two fixed points $x_1\in(U_2\setminus U_1)\cap\Omega$ and $x_2\in(\overline{U}_3\setminus\overline{U}_2)\cap\Omega.$ □

2 Integral formulation of some boundary value problems

2.1 First order boundary value problems

Consider the following first order differential equation

$$x' = f(t,x), \quad t \in [a,b], \tag{2.1.1}$$

subject to the boundary conditions

$$Mx(a) + Rx(b) = 0, \tag{2.1.2}$$

where $M, R \in \mathbb{R}$, $M + R \neq 0$, $a < b < \infty$ are given constants and $f \in \mathcal{C}([a,b] \times \mathbb{R})$.

Lemma 2.1.1. *The first order BVP (2.1.1)-(2.1.2) is equivalent to the following integral equation*

$$x(t) = \int_a^t f(s,x(s))ds - \frac{R}{M+R}\int_a^b f(s,x(s))ds, \quad t \in [a,b]. \tag{2.1.3}$$

Proof. Let $x : [a,b] \to \mathbb{R}$ satisfy (2.1.1) and (2.1.2). It is easy to see that

$$x\,(t) = x\,(a) + \int_a^t f(s,x(s))ds, \quad t \in [a,b]. \tag{2.1.4}$$

So (2.1.2) gives

$$0 = Mx(a) + R\left(x(a) + \int_a^b f(s, x(s))ds\right), \tag{2.1.5}$$

according to (2.1.5), we obtain

$$x(a) = -\frac{R}{M+R}\int_a^b f(s, x(s))ds. \tag{2.1.6}$$

So substituting (2.1.6) into (2.1.4), we obtain

$$x(t) = \int_a^t f(s, x(s))ds - \frac{R}{M+R}\int_a^b f(s, x(s))ds, \quad t \in [a, b].$$

Conversely, if x is a solution to (2.1.3) then is it easy to show that (2.1.1) and (2.1.2) hold by direct calculation. □

2.2 Second order two-point boundary value problems

We consider the following linear second order differential equation

$$(\mathcal{E}) \quad (p(t)y')' + q(t)y = h(t), \quad x \in (a, b),$$

where p, q et h are regular functions, subjected to separated linear boundary conditions :

$$(\mathfrak{F}) \begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = \gamma \\ \beta_1 y(b) + \beta_2 y'(b) = \delta, \end{cases}$$

where $\alpha_1^2 + \alpha_2^2 \neq 0$ et $\beta_1^2 + \beta_2^2 \neq 0$. In this case, The Green's function associated to the problem $(\mathcal{E}) + (\mathfrak{F})$ can be determined as follows :

$$G(t, s) = \frac{1}{p(t)W(t)} \begin{cases} \phi_1(t)\phi_2(s), & a \leq t \leq s \leq b, \\ \phi_1(s)\phi_2(t), & a \leq s \leq t \leq b, \end{cases}$$

where ϕ_1 and ϕ_2 are, respectively, the solutions of the Cauchy's problems:

$$(\mathcal{E}_H)+\begin{cases}\phi_1(a)=\alpha_2\\ \phi_1'(a)=-\alpha_1\end{cases} \quad \text{and} \quad (\mathcal{E}_H)+\begin{cases}\phi_2(b)=\beta_2\\ \phi_2'(b)=-\beta_1,\end{cases}$$

and $W(t)=\phi_1(t)\phi_2'(t)-\phi_1'(t)\phi_2(t)\neq 0$ is their Wronskian.

Note that the product pW is constant in $[a,b]$.

Particular case:

Consider the Dirichlet's problem posed in $[a,b]$

$$(\mathcal{P})\begin{cases}y''=h(t),\ a<t<b\\ y(a)=y(b)=0.\end{cases}$$

Let ϕ_1 and ϕ_2 be the solutions of Cauchy's problems :

$$\begin{cases}\phi_1''=0\\ \phi_1(a)=0\\ \phi_1'(a)=-1.\end{cases} \quad \text{and} \quad \begin{cases}\phi_2''=0\\ \phi_2(b)=0\\ \phi_2'(b)=-1.\end{cases}$$

We find $\phi_1(x)=a-t,\ \phi_2(x)=b-t,\ \ W(\phi_1,\phi_2)=b-a\neq 0$ and $p(t)=1,\ t\in[a,b]$.

Hence the Green's function

$$G(t,s)=\frac{1}{b-a}\begin{cases}(t-a)(s-b),\ \text{if } a\le t\le s\le b\\ (s-a)(t-b),\ \text{if } a\le s\le t\le b.\end{cases} \tag{2.2.1}$$

Lemma 2.2.1. *(The nonlinear case).*

Let $f\in\mathcal{C}([a,b]\times\mathbb{R})$, then the nonlinear boundary value problem

$$\begin{cases}-y''+f(t,y)=0, \quad t\in(a,b),\\ y(a)=0, \quad y(b)=0,\end{cases} \tag{2.2.2}$$

is equivalent to the integral equation

$$y(t)=\int_a^b G(t,s)\,f(s,y(s))ds,\ t\in[a,b].$$

where G is given in (2.2.1).

2.3 Second order three-point boundary value problems

Consider the following second order nonlinear differential equation

$$y'' + f(t, y) = 0, \quad t \in (a, b), \tag{2.3.1}$$

subject to the boundary conditions

$$y(a) = ky(\eta), \quad y(b) = 0, \tag{2.3.2}$$

where $\eta \in (0, 1)$, $k \in \mathbb{R}$ with $k(b - \eta) \neq b - a$ and $f \in \mathcal{C}([a, b] \times [0, \infty))$.

Lemma 2.3.1. *(The linear case, see[49]) Assume that $k\,(b - \eta) \neq b - a$ and $g \in \mathcal{C}([a, b])$. Then $y \in \mathcal{C}^2([a, b])$ is a solution of the linear boundary value problem*

$$\begin{cases} y'' + g(t) = 0, \quad t \in (a, b), \\ y(a) = ky(\eta), \quad y(b) = 0, \end{cases} \tag{2.3.3}$$

if and only if

$$y(t) = \int_a^b G(t, s)g(s)\, ds, \quad t \in [a, b], \tag{2.3.4}$$

where the Green's function G is defined on $[a, b] \times [a, b]$ by

$$G(t, s) = H(t, s) + \frac{k(b - t)}{b - a - k(b - \eta)} H(\eta, s), \quad t, s \in [a, b],$$

with

$$H(t, s) = \frac{1}{b - a} \begin{cases} (t - a)\,(b - s)\,, \quad a \leq t \leq s \leq b, \\ (s - a)\,(b - t)\,, \quad a \leq s \leq t \leq b. \end{cases} \tag{2.3.5}$$

Proof. It is well known that the function H, as in (2.3.5), is the Green function of the second-order two-point linear boundary value problem

$$\begin{cases} y'' + g(t) = 0, \quad t \in (a,b), \\ y(a) = 0, \quad y(b) = 0, \end{cases} \tag{2.3.6}$$

and the solution of (2.3.6) is given by

$$z(t) = \int_a^b H(t,s)\, g(s) ds, \quad t \in [a,b]. \tag{2.3.7}$$

Next, the three-point boundary value problem (2.3.3) can be obtained from replacing $y(a) = 0$ by $y(a) = ky(\eta)$ in (2.3.6). Thus, we suppose that the solution of the three-point boundary value problem (2.3.3) can be expressed by:

$$y(t) = z(t) + (c + d\,t)\, z(\eta),\ t \in [a,b], \tag{2.3.8}$$

where c and d are constants that will be determined.

From (2.3.7) and (2.3.8), we get

$$\begin{aligned} y(a) &= (c + d\,a)\, z(\eta); \\ y(b) &= (c + d\,b)\, z(\eta); \\ y(\eta) &= (c + d\,\eta + 1)\, z(\eta). \end{aligned}$$

Putting these into (2.3.2) yields

$$\begin{cases} c + d\,a = k\,(c + d\eta + 1)\,, \\ c + d\,b = 0. \end{cases} \tag{2.3.9}$$

Since $k\,(b - \eta) \neq b - a$, solving the linear system (2.3.9), we obtain

$$c = \tfrac{kb}{b-a-k(b-\eta)},$$
$$d = \tfrac{-k}{b-a-k(b-\eta)}.$$

Hence, $c+dt=\frac{k(b-t)}{b-a-k(b-\eta)}$. By substitution in (2.3.8), we deduce

$$y\left(t\right)=z\left(t\right)+\frac{k(b-t)}{b-a-k(b-\eta)}z\left(\eta\right).$$

This together with (2.3.7) implies that

$$y(t)=\int_a^b H(t,s)\,g(s)ds+\frac{k(b-t)}{b-a-k(b-\eta)}\int_a^b H(\eta,s)\,g(s)ds.$$

whence the form of the Green's function G.

Conversely, let $y\in\mathcal{C}^1([a,b])$ be defined by (2.3.4). A direct differentiation of (2.3.4) gives

$$y'(t)=\int_a^b G_t(t,s)g(s)\,ds,\qquad t\in[a,b], \tag{2.3.10}$$

where $G_t(t,s)=H_t(t,s)-\frac{k}{b-a-k(b-\eta)}\,H(\eta,s)$ is the partial derivative of $G(t,s)$ with respect to t and

$$H_t(t,s)=\frac{1}{b-a}\begin{cases} b-s, & a\le t\le s\le b,\\ a-s, & a\le s\le t\le b.\end{cases}$$

Differentiating again (2.3.10), we finally arrive at

$$y''(t)\;=\;-g(s),\qquad t\in(a,b).$$

Hence $y\in\mathcal{C}^2([a,b])$ and y satisfies (2.3.6). □

Consequently, we have

Corollary 2.3.2. *(The nonlinear case). Let $\eta\in(0,1)$, $k\in\mathbb{R}$ with $k(b-\eta)\neq b-a$ and $f\in\mathcal{C}([a,b]\times[0,\infty))$, then the nonlinear boundary value problem*

$$\begin{cases} y''+f(t,y)=0, & t\in[a,b],\\ y(a)=ky(\eta), & y(b)=0,\end{cases} \tag{2.3.11}$$

is equivalent to the integral equation

$$y(t)=\int_a^b G(t,s)\,f(s,y(s))ds,\ t\in(a,b).$$

where G is given by (2.3.1) in Lemma 2.3.1.

3 Existence of solutions for a class of first order boundary value problems

The results of this chapter are obtained by Mouhous, Goergiev and Mebarki in [36].

3.1 Introduction

In this chapter, we investigate the existence of solutions of the following first order differential equation

$$x' = f(t,x), \quad t \in [a,b], \tag{3.1.1}$$

subject to the boundary conditions

$$Mx(a) + Rx(b) = 0, \tag{3.1.2}$$

where $M, R \in \mathbb{R}$, $M + R \neq 0$, $a < b < \infty$ are given constants and

(H1) $f \in \mathcal{C}([a,b] \times \mathbb{R})$, $|f(t,x)| \leq \sum_{j=1}^{k} a_j(t)|x|^{p_j}$, $(t,x) \in [a,b] \times \mathbb{R}$, $a_j \in \mathcal{C}([a,b])$, $0 \leq a_j \leq A$ on $[a,b]$, $p_j \geq 0$, $j \in \{1,\ldots,k\}$.

The first-order BVPs arise in many applications of science, engineering and technology (see [5, Chapter 1]). Thanks to these applications, more theoretical studies of the subject can be developed, including: solvability, uniqueness, positivity and multiplicity of solutions. For the recent developments involving existence of solutions to BVPs for first order differential equations, we can refer to [18, 31, 39, 43, 44, 45, 46].

In this work, we propose a new approach to ensure the existence of solutions for the first order BVP (3.1.1)-(3.1.2). Our method involves new fixed point theorem (Theorem 1.3.3) for the sum of two operators. The problem (3.1.1)-(3.1.2) one can consider as a scalar-valued analogue of the problem in [44]. The scalar-valued analogues of the conditions used in [44] are as follows:

(**C1**) there exist nonnegative constants α and K so that

$$|f(t,x)| \leq \alpha x f(t,x) + K, \quad (t,x) \in [a,b] \times \mathbb{R}, \quad \text{and} \quad \left|\frac{M}{R}\right| \leq 1,$$

(**C2**) there exist nonnegative constants α and K so that

$$|f(t,x)| \leq -\alpha x f(t,x) + K, \quad (t,x) \in [a,b] \times \mathbb{R} \quad \text{and} \quad \left|\frac{R}{M}\right| \leq 1,$$

(**C3**) there exists a $\mathcal{C}^1$ function $V : \mathbb{R} \to [0,\infty)$ and nonnegative constants α and K so that

$$|f(t,x)| \leq \alpha V'(x) f(t,x) + K, \quad (t,x) \in [a,b] \times \mathbb{R} \quad \text{and} \quad V(x(a)) \geq V(x(b)),$$

(**C4**) there exists a $\mathcal{C}^1$ function $V : \mathbb{R} \to [0,\infty)$ and nonnegative constants α and K so that

$$|f(t,x)| \leq -\alpha V'(x) f(t,x) + K, \quad (t,x) \in [a,b] \times \mathbb{R} \quad \text{and} \quad V(x(a)) \leq V(x(b)).$$

Note that the conditions (C1),(C2), (C3),(C4) in the scalar-valued case are different from the condition $(H1)$. Moreover, in [44] there is an additional restriction $\left|\frac{M}{R}\right| \leq 1$ $\left(\left|\frac{R}{M}\right| \leq 1\right)$ on R and M. Thus, we can consider our main result as a complementary result to these of [44] in the scalar-valued case. Moreover, our main results are valid in the case when $R = 0$. Thus, our main results can be applied for the classical initial value problems of first order ODEs whenever f satisfies $(H1)$.

3.2 Main results

3.2.1 Auxiliary results

In [44], it is proved that the problem (3.1.1)-(3.1.2) is equivalent to the following integral equation

$$x(t) = \int_a^t f(s, x(s))ds - \frac{R}{M+R}\int_a^b f(s, x(s))ds, \quad t \in [a, b]. \tag{3.2.1}$$

Let $E = \mathcal{C}([a, b])$ be endowed with the maximum norm

$$\|x\| = \max_{t\in[a,b]} |x(t)|.$$

For $x \in E$, define the operator

$$S_1x(t) = \int_a^t f(s, x(s))ds - \frac{R}{M+R}\int_a^b f(s, x(s))ds - x(t), \quad t \in [a, b].$$

By (3.2.1), it follows that if $x \in E$ satisfies the equation $S_1x = 0$, then it is a solution to the problem (3.1.1)-(3.1.2). Fix $B > 0$ arbitrarily.

Lemma 3.2.1. *Suppose that* $(H1)$ *holds. For any* $x \in E$ *with* $\|x\| \le B$*, we have*

$$\|S_1x\| \le A\left(1 + \left|\frac{R}{M+R}\right|\right)(b-a)\sum_{j=1}^{k} B^{p_j} + B.$$

Proof. We have

$$\begin{aligned}
|S_1x(t)| &= \left|\int_a^t f(s, x(s))ds - \frac{R}{M+R}\int_a^b f(s, x(s))ds - x(t)\right| \\
&\le \int_a^b |f(s, x(s))|ds + \left|\frac{R}{M+R}\right|\int_a^b |f(s, x(s))|ds + |x(t)| \\
&\le \left(1 + \left|\frac{R}{M+R}\right|\right)\int_a^b \sum_{j=1}^{k} a_j(s)|x(s)|^{p_j}ds + B \\
&\le A\left(1 + \left|\frac{R}{M+R}\right|\right)(b-a)\sum_{j=1}^{k} B^{p_j} + B, \quad t \in [a, b],
\end{aligned}$$

whereupon

$$\|S_1x\| \leq A\left(1+\left|\frac{R}{M+R}\right|\right)(b-a)\sum_{j=1}^{k} B^{p_j} + B.$$

This completes the proof. □

Let $g \in \mathcal{C}([a,b])$ be positive except at a finite number of points on $[a,b]$ and

$$C = \int_a^b g(t)dt. \tag{3.2.2}$$

For $x \in E$, define the operator

$$S_2x(t) = \int_a^t g(\tau)S_1x(\tau)d\tau, \quad t \in [a,b].$$

Lemma 3.2.2. *Suppose* $(H1)$*. If* $x \in E$ *satisfies the integral equation*

$$S_2x(t) = c, \quad t \in [a,b],\ c \in \mathbb{R}, \tag{3.2.3}$$

then x *is a solution to the proplem* (3.1.1)-(3.1.2)*.*

Proof. We differentiate the equation (3.2.3) with respect to t and we get

$$g(t)S_1x(t) = 0, \quad t \in [a,b],$$

whereupon

$$S_1x(t) = 0, \quad t \in [a,b].$$

This completes the proof. □

Lemma 3.2.3. *Suppose that* $(H1)$ *hold. Let* $x \in E$ *be such that* $\|x\| \leq B$*. Then*

$$\|S_2x\| \leq C\left(A\left(1+\left|\frac{R}{M+R}\right|\right)(b-a)\sum_{j=1}^{k} B^{p_j} + B\right).$$

Proof. Using Lemma 3.2.1, we arrive at

$$\begin{aligned} |S_2x(t)| &= \left|\int_a^t g(\tau)S_1x(\tau)d\tau\right| \\ &\leq C\left(A\left(1+\left|\frac{R}{M+R}\right|\right)(b-a)\sum_{j=1}^{k} B^{p_j}+B\right), \quad t\in[a,b]. \end{aligned}$$

Hence,

$$\|S_2x\| \leq C\left(A\left(1+\left|\frac{R}{M+R}\right|\right)(b-a)\sum_{j=1}^{k} B^{p_j}+B\right).$$

This completes the proof. □

3.2.2 Existence of at least two nonnegative solutions

Let $m>0$ be large enough and A, r, L, R_1 be positive constants that satisfy the following inequalities

$$\textbf{(H2)} \quad \begin{cases} r<L<R_1, \quad R_1>\left(\frac{2}{5m}+1\right)L, \\ C\left(A\left(1+\left|\frac{R}{M+R}\right|\right)(b-a)\sum\limits_{j=1}^{k} R_1^{p_j}+R_1\right)<\frac{L}{5}, \end{cases}$$

where C is the constant which appears in (3.2.2). Let $\epsilon>0$, For $x\in E$, define the operators

$$\begin{aligned} T_1x(t) &= (1+m\epsilon)x(t)-\epsilon\frac{L}{10}, \\ S_3x(t) &= -\epsilon S_2x(t)-m\epsilon x(t)-\epsilon\frac{L}{10}, \quad t\in[a,b]. \end{aligned}$$

Note that any fixed point $x\in E$ of the operator $T+S_3$ is a solution to the problem (3.1.1)-(3.1.2). Our main result in this section is as follows.

Theorem 3.2.4. *Suppose that* $(H1)$ *and* $(H2)$ *hold. Then the problem* (3.1.1)-(3.1.2) *has at least two nontrivial nonnegative solutions in* $\mathcal{C}^1([a,b])$.

Proof. We will use Theorem 1.3.3. Define the positive cone

$$\tilde{\mathcal{P}} = \{x \in E : x \geq 0 \quad \text{on} \quad [0,1]\}.$$

With $\mathcal{P}$ we will denote the set of all equicontinuous families in $\widetilde{P}$. Let

$$\begin{aligned}
U_1 &= \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}, \\
U_2 &= \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}, \\
U_3 &= \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}, \\
R_2 &= R_1 + \frac{C}{m}(A(1 + |\frac{R}{M+R}|)(b-a)\sum_{j=1}^{k} R_1^{p_j} + R_1) + \frac{L}{5m}, \\
\Omega &= \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \leq R_2\}.
\end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have

$$\|T_1 v_1 - T_1 v_2\| = (1 + m\epsilon)\|v_1 - v_2\|,$$

whereupon $T_1 : \Omega \to E$ is an expansive operator with a constant $1 + m\epsilon$.

2. For $v \in \overline{\mathcal{P}_{R_1}}$, we get

$$\begin{aligned}
\|S_3 v\| &\leq \epsilon\|S_2 v\| + m\epsilon\|v\| + \epsilon\frac{L}{10} \\
&\leq \epsilon\Bigg(C\left(A\left(1 + \left|\frac{R}{M+R}\right|\right)(b-a)\sum_{j=1}^{k} R_1^{p_j} + R_1\right) + mR_1 + \frac{L}{10}\Bigg).
\end{aligned}$$

Therefore $S_3(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_{R_1}})$ is equicontinuous. Consequently $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is a completely continuous mapping.

3. Let $v_1 \in \overline{\mathcal{P}_{R_1}}$. Set

$$v_2 = v_1 + \frac{1}{m}S_2 v_1 + \frac{L}{5m}.$$

Note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $[a, b]$. We have $v_2 \geq 0$ on $[a, b]$ and

$$\begin{aligned}
\|v_2\| &\leq \|v_1\| + \frac{1}{m}\|S_2 v_1\| + \frac{L}{5m} \\
&\leq R_1 + \frac{C}{m}\left(A\left(1 + \left|\frac{R}{M+R}\right|\right)(b-a)\sum_{j=1}^{k} R_1^{p_j} + R_1\right) + \frac{L}{5m} \\
&= R_2.
\end{aligned}$$

Therefore $v_2 \in \Omega$ and

$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon S_2 v_1 - \epsilon\frac{L}{10} - \epsilon\frac{L}{10}$$

or

$$\begin{aligned}
(I - T_1)v_2 &= -\epsilon m v_2 + \epsilon\frac{L}{10} \\
&= S_3 v_1.
\end{aligned}$$

Consequently $S_3(\overline{\mathcal{P}_{R_1}}\,) \subset (I - T_1)(\Omega)$.

4. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda > 0$ and $x \in \partial\mathcal{P}_r \cap (\Omega + \lambda u_0)$ or $x \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda u_0)$ such that

$$S_3 x = (I - T_1)(x - \lambda u_0).$$

Then

$$-\epsilon S_2 x - m\epsilon x - \epsilon\frac{L}{10} = -m\epsilon(x - \lambda u_0) + \epsilon\frac{L}{10}$$

or

$$-S_2 x = \lambda m u_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 x\| = \left\|\lambda m u_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

This is a contradiction.

5. Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $\lambda_1 \geq \varepsilon_1 + 1$ and $x_1 \in \partial\mathcal{P}_L$, $\lambda_1 x_1 \in \overline{\mathcal{P}_{R_2}}$ such that

$$S_3 x_1 = (I - T_1)(\lambda_1 x_1). \tag{3.2.4}$$

Since $x_1 \in \partial\mathcal{P}_L$ and $\lambda_1 x_1 \in \overline{\mathcal{P}_{R_2}}$, it follows that

$$\left(\frac{2}{5m} + 1\right) L < \lambda_1 L = \lambda_1 \|x_1\| \leq R_2.$$

Moreover, $-\epsilon S_2 x_1 - m\epsilon x_1 - \epsilon \frac{L}{10} = -\lambda_1 m\epsilon x_1 + \epsilon \frac{L}{10}$,

or

$$S_2 x_1 + \frac{L}{5} = (\lambda_1 - 1) m x_1.$$

From here,

$$2\frac{L}{5} > \left\| S_2 x_1 + \frac{L}{5} \right\| = (\lambda_1 - 1) m \|x_1\| = (\lambda_1 - 1) m L$$

and

$$\frac{2}{5m} + 1 > \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 1.3.3 hold. Hence, the problem (3.1.1)-(3.1.2) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

This completes the proof. □

3.2.3 Applications

Consider the boundary value problem:

$$\begin{aligned} x'(t) &= (x(t))^2 + \tfrac{1}{1+t^2}(x(t))^4 + 1, \quad t \in [0,1], \\ 2x(0) + x(1) &= 0. \end{aligned} \tag{3.2.5}$$

Here

$$\begin{aligned} f(t,x) &= x^2 + \frac{1}{1+t^2}x^4 + 1, \quad k = 3, \quad a_1(t) = 1, \quad a_2(t) = \frac{1}{1+t^2}, \quad t \in [0,1], \\ a &= 0, \quad b = 1, \quad p_1 = 2, \quad p_2 = 4, \quad p_3 = 0, \quad M = 2, \quad R = 1. \end{aligned}$$

Firstly, we will note that the scalar-valued case of the results in [44] are not applicable for the BVP (3.2.5). Here $\frac{R}{M} = \frac{1}{2} < 1$. Assume that there are nonnegative constants α and K so that $|f(t,x)| \leq -\alpha x f(t,x) + K, \quad (t,x) \in [0,1] \times \mathbb{R}$, which is equivalent to

$$x^2 + \frac{1}{1+t^2}x^4 + 1 \leq -\alpha x\left(x^2 + \frac{1}{1+t^2}x^4 + 1\right) + K, \quad (t,x) \in [0,1] \times \mathbb{R},$$

or

$$(1+\alpha x)\left(x^2 + \frac{1}{1+t^2}x^4 + 1\right) \leq K, \quad (t,x) \in [0,1] \times \mathbb{R}.$$

The last inequality is impossible because

$$\lim_{x\to\infty}(1+\alpha x)\left(x^2 + \frac{1}{1+t^2}x^4 + 1\right) = \infty,$$

i.e., $(C2)$ does not hold. Now, we will show that our main results are applicable for the BVP (3.2.5). We have $A = 1$. Take $B = 1$. We have

$$A\left(1 + \left|\frac{R}{M+R}\right|\right)(b-a)\sum_{j=1}^{k} B^{p_j} + B = \left(1 + \frac{1}{3}\right)(1+1+1) + 1 = 5.$$

Let

$$g(t) = \frac{2}{10^{10}}t, \quad t \in [0,1].$$

Then

$$\int_0^1 g(t)dt = \frac{2}{10^{10}} \int_0^1 tdt = \frac{1}{10^{10}}.$$

Take $C = \epsilon = \frac{1}{10^{10}}$. Let now

$$R_1 = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}.$$

Then

$$r < L < R_1, \quad 10 = R_1 > \left(\frac{2}{5 \cdot 10^{50}} + 1\right) 5 = \left(\frac{2}{5m} + 1\right) L$$

and

$$C \left(A \left(1 + \left|\frac{R}{M+R}\right|\right) (b-a) \sum_{j=1}^{k} R_1^{p_j} + R_1 \right)$$

$$= \frac{1}{10^{10}} \left(\frac{4}{3} \cdot \left(10^2 + 10^3 + 1\right) + 10\right) < \frac{1}{10^5} < 1 = \frac{L}{5}.$$

So, $(H2)$ holds. Then, by Theorem 3.2.4, it follows that the BVP (3.2.5) has at least two nonnegative solutions.

Let now, $R = 0$ and f, k, a_1, a_2, a, b, p_1, p_2, p_3, M, R_1, L, r, m, C, ϵ and g be as above. Consider the IVP

$$\begin{aligned} x'(t) &= (x(t))^2 + \tfrac{1}{1+t^2} (x(t))^4 + 1, \quad t \in [0,1], \\ x(0) &= 0. \end{aligned} \tag{3.2.6}$$

Then

$$A \left(1 + \left|\frac{R}{M+R}\right|\right) (b-a) \sum_{j=1}^{k} B^{p_j} + B = 1 \cdot (1+1+1) + 1 = 4$$

$$C \left(A \left(1 + \left|\frac{R}{M+R}\right|\right) (b-a) \sum_{j=1}^{k} R_1^{p_j} + R_1 \right)$$
$$= \frac{1}{10^{10}} \left(1 \cdot \left(10^2 + 10^3 + 1\right) + 10\right) < \frac{1}{10^5} < 1 = \frac{L}{5}.$$

So, $(H2)$ holds. Then the IVP (3.2.6) has at least two nonnegative solutions.

3.3 Concluding remarks

1. In this work, we have investigated a class of boundary value problems for first order ODEs. The nonlinear term depends on the solution and may change sign, and it satisfies general polynomial growth conditions. We prove existence of at least one solution and two nonnegative solutions in $\mathcal{C}^1([a,b])$. The proof of the main results is based upon recent theoretical results.

2. The conditions (3.1.2) are general and they capture in particular the anti-periodic conditions corresponding to the case $M = R = 1$.

3. Our main results obtained in this work and the results in [44] are complementary. Moreover, in [44], there is an additional restriction on the constants R and M $(\left|\frac{M}{R}\right| \leq 1$ or $\left|\frac{R}{M}\right| \leq 1)$. Note that our main results also depend on the hypothesis $(H2)$, where the conditions are controlled by the constants C, ϵ and B and the source term f does not depend on these constants.

4. New existence results of multiple non trivial nonnegative solutions are proved using recent fixed point theorems on cones in Banach spaces for the sum of two operators.

5. It is noted that Theorem 3.2.4 can be generalized to the case where $f \in \mathcal{C}([a,b] \times \mathbb{R}^n, \mathbb{R}^n)$, $n > 1$. In this case, we will consider the space $E_1 = (\mathcal{C}([a,b]))^n$ endowed with the norm

$$\|x\|_1 = \max_{j\in\{1,\ldots,n\}} \|x_j\|, \quad x = (x_1, \ldots, x_n).$$

The hypothesis $(H1)$ takes the form

(H1′) $f \in \mathcal{C}([a,b] \times \mathbb{R}, \mathbb{R}^n)$, $f = (f_1, \ldots, f_n)$, $|f_i(t,x)| \leq \sum\limits_{j=1}^{k} a_{ji}(t)|x|^{p_{ji}}$,

$(t, x) \in [a, b] \times \mathbb{R}^n$, $a_{ji} \in \mathcal{C}([a, b])$, $0 \leq a_{ji} \leq A$ on $[a, b]$, $p_{ji} \geq 0$, $j \in \{1, \ldots, k\}$, $i \in \{1, \ldots, n\}$,

and the hypothesis $(H2)$ will be the same.

6. These theoretical results can be used to study other classes of BVP as well as some IVP in ODEs. For these aims, firstly we search an integral representation of the solutions of the considered IVPs/BVPs. Then we use it to define the operators S_1, S_2, S, $\widetilde{S}$ and T. Finally, we apply Theorem 1.3.3.

4 Existence of fixed points in conical shells of a Banach space and application to ODEs

The results of this chapter are obtained by Mouhous and Mebarki in [35].

4.1 Introduction

In this chapter, the functional expansion-compression fixed point theorem of Leggett-Williams type developed in [3] is extended to the class of mappings of the form $T + F$, where $(I - T)$ is Lipschitz invertible and F is a k-set contraction. As application, the existence and multiplicity of nontrivial nonnegative solutions for a nonlinear second order three-point boundary value problem is established.

Recently, in 2019 a new direction of research in the theory of fixed point in ordered Banach spaces for the sum of two operators is opened by Djebali and Mebarki [16]. Then, several fixed point theorems, including Krasnosel'skii and Leggett-Williams types theorems in cones, have been established (see [11, 14, 15, 20, 22, 26]). These theorems have been applied to obtain existence results for nonnegative solutions of various types of boundary and/or initial value problems (see [19, 20, 23, 24, 26]). In our work, we have used the fixed point index theory developed in [16] and [26] to generalize the main result of [3, Theorem 4.1] for the sum $T+F$ where $(I-T)$ is a Lipschitz invertible mapping with constant $\gamma>0$ and F is a k-set contraction with $k\,\gamma<1$.

4.2 Main results

Let $\mathcal{P}$ be a cone in a Banach space $(E,\|.\|)$ and Ω a subset of $\mathcal{P}$. Our main result is as follows.

Theorem 4.2.1. *Let α be a nonnegative continuous concave functional on $\mathcal{P}$ and β be a nonnegative continuous convex functional on $\mathcal{P}$. Let $T:\Omega\subset\mathcal{P}\to E$ be such that $(I-T)$ is Lipschitz invertible mapping with constant $\gamma>0$, $F:\mathcal{P}\to E$ is a k-set contraction with $0\le k<\gamma^{-1}$. Assume that there exist four nonnegative numbers a, b, c, d and $z_0\in\mathcal{P}$ such that $\beta((I-T)^{-1}0)<b$, $\alpha((I-T)^{-1}z_0)>c$ and*

$$Fx+Tx\in\mathcal{P},\;\; Tx\in\mathcal{P},\quad \text{for all } x\in\partial\mathcal{P}(\beta,b)\cup\partial\mathcal{P}(\alpha,c),$$

$$\lambda\,F(\mathcal{P}(\beta,b))\subset(I-T)(\Omega),\;\text{for all } \lambda\in[0,1], \tag{4.2.1}$$

$$\lambda\,F(\mathcal{P}(\alpha,c))+(1-\lambda)z_0\subset(I-T)(\Omega),\;\text{for all } \lambda\in[0,1]. \tag{4.2.2}$$

Suppose that:

($\mathcal{A}1$) *if $x\in\mathcal{P}$ with $\beta(x)=b$, then $\alpha(Tx)\ge a$;*

($\mathcal{A}2$) *if $x\in\mathcal{P}$ with $\beta(x)=b$ and $[\alpha(x)\ge a$ or $\alpha(Tx+Fx)<a]$, then $\beta(Tx+Fx)<b$ and $\beta(Tx)\le b$;*

($\mathcal{A}3$) *if $x\in\mathcal{P}$ with $\alpha(x)=c$, then $\beta(Tx+z_0)\le d$;*

($\mathcal{A}4$) *if $x\in\mathcal{P}$ with $\alpha(x)=c$ and $[\beta(x)\le d$ or $\beta(Tx+Fx)>d]$, then $\alpha(Tx+Fx)>c$ and $\alpha(Tx+z_0)\ge c$;*

Then,

1. *(Expansive form)* $T + F$ *has a fixed point* x^* *in* $\mathcal{P}(\beta, \alpha, b, c) \cap \Omega$ *if*

   ($\mathcal{H}1$) $a < c$, $b < d$, $\{x \in \mathcal{P} : b < \beta(x) \text{ and } \alpha(x) < c\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$, $\mathcal{P}(\beta, b) \cap \Omega \neq \emptyset$ *and* $\mathcal{P}(\alpha, c)$ *is bounded.*

2. *(Compressive form)* $T + F$ *has a fixed point* x^* *in* $\mathcal{P}(\alpha, \beta, c, b) \cap \Omega$ *if*

   ($\mathcal{H}2$) $c < a$, $d < b$, $\{x \in \mathcal{P} : c < \alpha(x) \text{ and } \beta(x) < b\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\alpha, c) \subset \mathcal{P}(\beta, b)$, $\mathcal{P}(\alpha, c) \cap \Omega \neq \emptyset$, *and* $\mathcal{P}(\beta, b)$ *is bounded.*

Proof. We will prove the expansion form. The proof of the compression form is similar.

We list

$$U = \{x \in \mathcal{P} : \beta(x) < b\},$$

$$V = \{x \in \mathcal{P} : \alpha(x) < c\}.$$

Then, the interior of $V - U$ is given by

$$W = (V - U)^o = \{x \in \mathcal{P} : \ b < \beta(x) \text{ and } \alpha(x) < c\}.$$

Thus U, V and W are bounded, not empty and open subsets of $\mathcal{P}$. To prove the existence of a fixed point for the sum $T + F$ in $\mathcal{P}(\beta, \alpha, b, c) \cap \Omega$, it is enough for us to show that $i_*(T + F, W \cap \Omega, \mathcal{P}) \neq 0$ since W is the interior of $\mathcal{P}(\beta, \alpha, b, c)$.

Claim 1. $Tx + Fx \neq x$ for all $x \in \partial U \cap \Omega$.

Let $x_0 \in \partial U \cap \Omega$, then $\beta(x_0) = b$. Suppose that $x_0 = Tx_0 + Fx_0$, then $\beta(Tx_0 + Fx_0) = b$. By the condition ($\mathcal{A}2$), if $\alpha(x_0) \geq a$, then $\beta(Tx_0 + Fx_0) < b$, and if $\alpha(x_0) < a$, thus $\alpha(Tx_0 + Fx_0) < a$, then $\beta(Tx_0 + Fx_0) < b$. This is a contradiction. Thus we have $Tx + Fx \neq x$ for all $x \in \partial U \cap \Omega$.

Claim 2. $Tx + Fx \neq x$ for all $x \in \partial V \cap \Omega$.

Let $x_1 \in \partial V \cap \Omega$, then $\alpha(x_1) = c$. Suppose that $x_1 = Tx_1 + Fx_1$, then $\alpha(Tx_1 + Fx_1) = c$. By the condition $(\mathcal{A}4)$, if $\beta(x_1) \leq d$, then $\alpha(Tx_1 + Fx_1) > c$, and if $\beta(x_1) > d$, thus $\beta(Tx_1 + Fx_1) > d$, then $\alpha(Tx_1 + Fx_1) > c$. This is a contradiction. Thus we have $Tx + Fx \neq x$ for all $x \in \partial V \cap \Omega$.

Claim 3. $i_*(T + F, U \cap \Omega, \mathcal{P}) = 1$.

Let $H_1 : [0, 1] \times \overline{U} \to E$ be defined by

$$H_1(t, x) = tFx.$$

Clearly H_1 is continuous and uniformly continuous in t with respect to $x \in \overline{U}$, and from (4.2.2) we easily see that $H_1([0, 1] \times \overline{U}) \subset (I - T)(\Omega)$. Moreover $H_1(t, .) : \overline{U} \to E$ is a k-set contraction for all $t \in [0, 1]$ and $Tx + H_1(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial U \cap \Omega$. Otherwise, there would exists $(t_2, x_2) \in [0, 1] \times \partial U \cap \Omega$ such that $Tx_2 + H_1(t_2, x_2) = x_2$. Since $x_2 \in \partial U$, $\beta(x_2) = b$. Either $\alpha(Tx_2 + Fx_2) < a$ or $\alpha(Tx_2 + Fx_2) \geq a$.

Case (1): If $\alpha(Tx_2 + Fx_2) < a$, the convexity of β and the condition $(\mathcal{A}2)$ lead

$$\begin{aligned} b = \beta(x_2) &= \beta\left(Tx_2 + H_1(t_2, x_2)\right) \\ &= \beta\left((1 - t_2)Tx_2 + t_2(Tx_2 + Fx_2)\right) \\ &\leq (1 - t_2)\beta(Tx_2) + t_2\beta(Tx_2 + Fx_2) \\ &< b, \end{aligned}$$

which is a contradiction.

Case (2): If $\alpha(Tx_2 + Fx_2) \geq a$, from the concavity of α and the condition $(\mathcal{A}1)$, we obtain $\alpha(x_2) \geq a$. Indeed,

$$\begin{aligned} \alpha(x_2) &= \alpha\left(Tx_2 + H_1(t_2, x_2)\right) \\ &\geq (1 - t_2)\alpha(Tx_2) + t_2\alpha(Tx_2 + Fx_2) \\ &\geq a, \end{aligned}$$

and thus by the condition $(\mathcal{A}2)$, we have $\beta(Tx_2 + Fx_2) < b$ and $\beta(Tx_2) < b$, which is the same contradiction we arrived at in the previous case.

Being $(I - T)^{-1}0 \in U \cap \Omega$, the homotopy invariance property (iii) and the normality property (i) of the index i_* lead

$$i_*(T + F, U \cap \Omega, \mathcal{P}) = i_*(T + 0, U \cap \Omega, \mathcal{P}) = 1.$$

Claim 4. $i_*(T + F, V \cap \Omega, \mathcal{P}) = 0.$

Let $H_2 : [0, 1] \times \overline{V} \to E$ be defined by

$$H_2(t, x) = t\, Fx + (1 - t)z_0.$$

Clearly H_2 is continuous and uniformly continuous in t with respect to $x \in \overline{V}$, and from (4.2.2) we easily see that $(H_2([0, 1] \times \overline{V})) \subset (I - T)(\Omega)$. Moreover $H_2(t, .) : \overline{V} \to E$ is a k-set contraction for all $t \in [0, 1]$ and $Tx + H_2(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial V \cap \Omega$. Otherwise, there would exists $(t_3, x_3) \in [0, 1] \times \partial V \cap \Omega$ such that $Tx_3 + H_2(t_3, x_3) = x_3$. Since $x_3 \in \partial V$ we have that $\alpha(x_3) = c$. Either $\beta(Tx_3 + Fx_3) \leq d$ or $\beta(Tx_3 + Fx_3) > d$.

Case (1): If $\beta(Tx_3 + Fx_3) > d$. the concavity of α and the condition $(\mathcal{A}4)$ lead

$$\begin{aligned} c = \alpha(x_3) &= \alpha(Tx_3 + H_2(t_3, x_3)) \\ &= \alpha(t_3(Tx_3 + Fx_3) + (1 - t_3)(Tx_3 + z_0)) \\ &\geq t_3\alpha(Tx_3 + Fx_3) + (1 - t_3)\alpha(Tx_3 + z_0) \\ &> c. \end{aligned}$$

This is a contradiction.

Case (2): If $\beta(Tx_3 + Fx_3) \leq d$, from the convexity of β and the condition $(\mathcal{A}3)$, we obtain $\beta(x_3) \leq d$. Indeed,

$$\begin{aligned} \beta(x_3) &= \beta(Tx_3 + H_2(t_3, x_3)) \\ &\leq t_3\beta(Tx_3 + Fx_3) + (1 - t_3)\beta(Tx_3 + z_0) \\ &\leq d, \end{aligned}$$

and thus by the condition $(\mathcal{A}4)$, we have $\alpha(Tx_3 + Fx_3) > c$, this is the same contradiction that we found in the previous case.

Hence, the homotopy invariance property (iii) of the fixed index i_* yields

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + z_0, V \cap \Omega, \mathcal{P}),$$

and by the solvability property (iv) of the index i_* (since $(I - T)^{-1}z_0 \notin V$ the index cannot be nonzero) we have

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + z_0, V \cap \Omega, \mathcal{P}) = 0.$$

Since U and W are disjoint open subsets of V and $T+F$ has no fixed points in $\overline{V}-(U \cup W)$ (by Claims 1 and 2), from the additivity property (ii) of the index i_*, we deduce

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + F, U \cap \Omega, \mathcal{P}) + i_*(T + F, W \cap \Omega, \mathcal{P}).$$

Consequently, we get

$$i_*(T + F, W \cap \Omega, \mathcal{P}) = -1,$$

and thus by the solvability property (iv) of the fixed point index i_*, the sum $T + F$ has a fixed point $x^* \in W \cap \Omega \subset \mathcal{P}(\beta, \alpha, b, c) \cap \Omega$. □

Now we add restrictions on the operator $T + F$ of Theorem 4.2.1 and we combine the expansive form and the compressive form to establish a multiplicity result.

Theorem 4.2.2. *Let α be a nonnegative continuous concave functional on $\mathcal{P}$ and β, γ be nonnegative continuous convex functionals on $\mathcal{P}$ for all $x \in \mathcal{P}$. Let $T : \Omega \subset \mathcal{P} \to E$ be such that $(I - T)$ is Lipschitz invertible mapping with constant $\zeta > 0$, $F : \mathcal{P} \to E$ is a k-set contraction with $0 \leq k < \zeta^{-1}$. Assume that there exist six nonnegative numbers $a < c < r$, $b < d < R$ and $z_0 \in \mathcal{P}$ such that*

$$\beta((I - T)^{-1}0) < b, \ \gamma((I - T)^{-1}0) < R, \ \ \alpha((I - T)^{-1}z_0) > c,$$

$$Fx + Tx \in \mathcal{P},\ Tx \in \mathcal{P},\quad \textit{for all } x \in \partial\mathcal{P}(\beta, b) \cup \partial\mathcal{P}(\alpha, c) \cup \partial\mathcal{P}(\gamma, R),$$

$$\lambda F(\mathcal{P}(\gamma, R)) \subset (I - T)(\Omega),\ \textit{for all } \lambda \in [0, 1],$$

$$\lambda F(\mathcal{P}(\alpha, c)) + (1 - \lambda)z_0 \subset (I - T)(\Omega),\ \textit{for all } \lambda \in [0, 1].$$

In addition to the assumptions $(\mathcal{A}1) - (\mathcal{A}4)$ *of Theorem 4.2.1, we suppose that the following conditions hold:*

($\mathcal{B}1$) *if* $x \in \mathcal{P}$ *with* $\gamma(x) = R$*, then* $\alpha(Tx) \geq r$*;*

($\mathcal{B}2$) *if* $x \in \mathcal{P}$ *with* $\gamma(x) = R$ *and* $[\alpha(x) \geq r$ *or* $\alpha(Tx + Fx) < r]$*, then* $\gamma(Tx + Fx) < R$ *and* $\gamma(Tx) \leq R$*.*

If the two following conditions hold,

($\mathbb{H}1$) $\{x \in \mathcal{P} : b < \beta(x)$ *and* $\alpha(x) < c\} \cap \Omega \neq \emptyset$*,* $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$*,* $\mathcal{P}(\beta, b) \cap \Omega \neq \emptyset$ *and* $\mathcal{P}(\alpha, c)$ *is bounded,*

($\mathbb{H}2$) $\{x \in \mathcal{P} : c < \alpha(x)$ *and* $\gamma(x) < R\} \cap \Omega \neq \emptyset$*,* $\mathcal{P}(\alpha, c) \subset \mathcal{P}(\gamma, R)$*,* $\mathcal{P}(\alpha, c) \cap \Omega \neq \emptyset$*, and* $\mathcal{P}(\gamma, R)$ *is bounded,*

then, $T + F$ *has at least two nontrivial fixed points* $x_1, x_2 \in \mathcal{P}$ *such that*

$$x_1 \in \mathcal{P}(\beta, \alpha, b, c) \cap \Omega \quad \textit{and} \quad x_2 \in \mathcal{P}(\alpha, \gamma, c, R) \cap \Omega.$$

Proof. We list

$$U = \{x \in \mathcal{P} : \beta(x) < b\},$$

$$V = \{x \in \mathcal{P} : \alpha(x) < c\},$$

$$Y = \{x \in \mathcal{P} : \gamma(x) < R\}.$$

Then, the interior of $V - U$ is given by

$$W = (V - U)^o = \{x \in \mathcal{P} :\ b < \beta(x) \text{ and } \alpha(x) < c\},$$

and the interior of $Y - V$ is given by

$$Z = (Y - V)^o = \{x \in \mathcal{P} : \ c < \alpha(x) \text{ and } \gamma(x) < R\}.$$

Thus U, V, Y and W, Z are bounded, not empty and open subsets of $\mathcal{P}$. To prove the existence of two fixed point for the sum $T+F$ in $\mathcal{P}(\beta, \alpha, b, c) \cap \Omega$ and $\mathcal{P}(\alpha, \gamma, c, R) \cap \Omega$ it is enough for us to show that $i_*(T+F, W \cap \Omega, \mathcal{P}) \neq 0$ and $i_*(T+F, Z \cap \Omega, \mathcal{P}) \neq 0$ since W is the interior of $\mathcal{P}(\beta, \alpha, b, c)$ and Z is the interior of $\mathcal{P}(\alpha, \gamma, c, R)$.

The use of the fixed point index here is similar to the proof of Theorem 4.2.1. □

4.3 Application to EDOs

In the sequel, we will investigate the three-point BVP:

$$\begin{aligned} &y'' + f(t, y) = 0, \quad t \in (0, 1), \\ &y(0) = ky(\eta), \quad y(1) = 0, \end{aligned} \tag{4.3.1}$$

where $\eta \in (0,1)$, $k > 0$ with $k(1-\eta) < 1$ and $f \in \mathcal{C}([0,1] \times [0, \infty))$. Set $B = \frac{1+k\eta}{1-k(1-\eta)}$ and suppose that

($\mathcal{C}$1) $\tilde{A} < f(t, y) \leq a_1(t) + a_2(t)|y|^p$ for $t \in [0,1]$ and $y \in [0, \infty)$, $a_1, a_2 \in \mathcal{C}([0,1])$, $0 \leq a_1, a_2 \leq A$ on $[0,1]$, for some positive constants $A, \tilde{A}$ and p.

($\mathcal{C}$2) $\epsilon \in (0,1)$, and there exist $a, b, c, d, z_0, \rho > 0$ such that

$$\max(d, \frac{2z_0}{\epsilon}, \frac{1}{\Lambda}(c - z_0)) < b \leq \rho; \quad 3z_0 > a;$$
$$z_0 \leq c < \min(a, 3\,z_0, \frac{\eta}{3}\left(1 - \tfrac{\eta}{2}\right)\tilde{A} + (1 - \tfrac{1}{\epsilon})z_0);$$
$$\frac{\epsilon AB(1 + b^p) + 3z_0}{\epsilon} \leq \rho; \quad (1-\epsilon)\tfrac{c}{\Lambda} + 3z_0 \leq d, \quad \text{where } \Lambda = \frac{\min\left(\epsilon\dfrac{\eta^2}{18}\left(1 - \dfrac{\eta}{2}\right)\tilde{A},\ z_0\right)}{\epsilon\,\rho},$$

and

$$AB(1 + b^p) < b. \tag{4.3.2}$$

Remark 4.3.1. *1. We end this section by an illustrative example, in which we give the constants $\epsilon, a, b, c, d, \rho, z_0$ and the function f that satisfy ($\mathcal{C}$1)-($\mathcal{C}$2). After setting the constants $A, \tilde{A}$ and p, we choose the constants $\epsilon, a, b, d, z_0, c$ and ρ.*

2. Discussion of Hypothesis (4.3.2):

(a) *If $p = 1$, the inequality (4.3.2) may be rewritten as $(\frac{1}{AB} - 1)\, b > 1$. A necessary condition for (4.3.2) to hold is that $A < \frac{1}{B}$.*

(b) *If $p \neq 1$, the inequality (4.3.2) can be written as $Kb - b^p > 1$ with $K = \frac{1}{AB}$. Consider the continuous function $\Phi(x) = Kx - x^p$ on $[0, \infty)$, then*

$$\Phi'(x) = 0 \Leftrightarrow x = x_0 = \sqrt[p-1]{\frac{K}{p}}.$$

(i) When $p < 1$, the function Φ verifies $\Phi(0) = 0$ and $\lim_{x \to +\infty} \Phi(x) = +\infty$. Moreover, Φ is decreasing on $[0, x_0)$ and increasing on (x_0, ∞) and assumes $\frac{K}{p} \sqrt[p-1]{\frac{K}{p}}(p-1)$ as a minimum at the point x_0. Hence for every real number $r > 0$, there exists a constant $b > 0$ with $\Phi(b) > r$. In particular $\Phi(b) > 1$.

(ii) When $p > 1$, the function Φ verifies $\Phi(0) = 0$ and $\lim_{x \to +\infty} \Phi(x) = -\infty$. Moreover, Φ is increasing on $[0, x_0)$ and decreasing on (x_0, ∞) and assumes $\frac{K}{p} \sqrt[p-1]{\frac{K}{p}}\,(p-1)$ as a maximum at $x = x_0$. Hence the inequality $\Phi(b) > 1$ has a solution $b > 0$ if and only if $\frac{K}{p} \sqrt[p-1]{\frac{K}{p}}\,(p-1) > 1$.

4.3.1 Existence of at least one nonnegative solution

Our existence result is as follows.

Theorem 4.3.2. *Suppose $(\mathcal{C}1)$ and $(\mathcal{C}2)$. Then the BVP (4.3.1) has at least one nontrivial nonnegative solution $y \in \mathcal{C}^2([0, 1])$ such that $c < \min_{t \in [\frac{\eta}{3}, \frac{\eta}{2}]} y(t) + z_0$ and $\max_{t \in [0,1]} |y(t)| < b$.*

Proof. To prove our main result, we will use Theorem 4.2.1.

Set

$$H(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

In [49] it is proved that the solution of the BVP (4.3.1) can be expressed in the following form

$$y(t) = \int_0^1 G(t,s)\, f(s, y(s))ds, \quad t \in [0,1],$$

where

$$G(t,s) = H(t,s) + \frac{k(1-t)}{1-k(1-\eta)} H(\eta, s), \quad t, s \in [0,1].$$

Note that $0 \le H(t,s) \le 1,\ \ t, s \in [0,1]$. Hence,

$$\begin{aligned} 0 \le G(t,s) \le 1 + \frac{k}{1-k(1-\eta)} &= \frac{1-k+k\eta+k}{1-k(1-\eta)} \\ &= \frac{1+k\eta}{1-k(1-\eta)} = B, \quad t, s \in [0,1]. \end{aligned}$$

Moreover, for $t, s \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]$, we have

$$H(t,s) \ge \frac{\eta}{3}\left(1 - \frac{\eta}{2}\right)$$

and

$$G(t,s) \ge H(t,s) \ge \frac{\eta}{3}\left(1 - \frac{\eta}{2}\right).$$

Next,

$$H_t(t,s) = \begin{cases} -s, & 0 \le s \le t \le 1, \\ 1-s, & 0 \le t \le s \le 1. \end{cases}$$

Hence, $|H_t(t,s)| \le 1$, $t, s \in [0,1]$, and

$$\begin{aligned} |G_t(t,s)| &= \left| H_t(t,s) - \frac{k}{1-k(1-\eta)} H(\eta,s) \right| \\ &\le |H_t(t,s)| + \frac{k}{1-k(1-\eta)} H(\eta,s) \\ &\le 1 + \frac{k}{1-k(1-\eta)} = \frac{1+k\eta}{1-k(1-\eta)} = B, \quad t, s \in [0,1]. \end{aligned}$$

Let $E = \mathcal{C}([0,1])$ be endowed with the maximum norm

$$\|y\| = \max_{t\in[0,1]} |y(t)|.$$

Define

$$\begin{aligned} \mathcal{P} &= \left\{ y \in E: \ y(t) \geq 0, \ \ t \in [0,1], \ \ \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) \geq \Lambda \|y\| \right\}, \\ \Omega &= \{ y \in \mathcal{P} : \|y\| \leq \rho \}. \end{aligned}$$

For $y \in \mathcal{P}$, let us define

$$\alpha(y) = \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) + z_0, \quad \beta(y) = \max_{t\in[0,1]} |y(t)|.$$

It's obvious that, since $\frac{2z_0}{\epsilon} < b \leq \rho$, we get $\Lambda < 1$.

For $y \in \mathcal{P}$, define the operators

$$\begin{aligned} Ty(t) &= (1-\epsilon) y(t) + 2z_0, \\ Fy(t) &= \epsilon \int_0^1 G(t,s) f(s, y(s)) ds - 2z_0, \quad t \in [0,1]. \end{aligned}$$

Note that if $y \in \mathcal{P}$ is a fixed point of the operator $T + F$, then it is a solution to the BVP (4.3.1). Next, if $y \in \mathcal{P}$ and $\|y\| \leq b$, we have

$$\begin{aligned} |Ty(t)| &\leq (1-\epsilon) y(t) + 2z_0 \\ &\leq (1-\epsilon) b + 2z_0 \\ &< b, \quad t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} |Ty(t) + Fy(t)| &= \left| (1-\epsilon) y(t) + \epsilon \int_0^1 G(t,s) f(s,y(s)) ds \right| \\ &\leq (1-\epsilon) y(t) + \epsilon \int_0^1 G(t,s) \left(a_1(s) + a_2(s) |y(s)|^p \right) ds \\ &\leq (1-\epsilon) b + \epsilon AB(1 + b^p) \\ &< b, \quad t \in [0,1]. \end{aligned}$$

Therefore, if $y \in \mathcal{P}$ and $\|y\| \le b$, we have

$$\|Ty\| < b, \tag{4.3.3}$$

and

$$\|Ty + Fy\| < b. \tag{4.3.4}$$

1. For $y, z \in \mathcal{P}$, we have

$$|(I - T)y(t) - (I - T)z(t)| = \epsilon |y(t) - z(t)|, \quad t \in [0, 1].$$

Hence,

$$\|(I - T)y - (I - T)z\| = \epsilon \|y - z\|.$$

Thus, $I - T : \mathcal{P} \to E$ is Lipschitz invertible operator with constant $\gamma = \frac{1}{\epsilon}$.

2. Let $y \in \mathcal{P}$. Then

$$\begin{aligned} |Fy(t))| &\le \epsilon \left| \int_0^1 G(t, s) f(s, y(s)) ds \right| + 2z_0 \\ &\le \epsilon AB(1 + \|y\|^p) + 2z_0, \quad t \in [0, 1], \end{aligned}$$

whereupon

$$\|Fy\| \le \epsilon AB(1 + \|y\|^p) + 2z_0 < \infty.$$

Moreover,

$$\begin{aligned} \left| \frac{d}{dt} Fy(t) \right| &= \left| \epsilon \int_0^1 G_t(t, s) f(s, y(s)) ds \right| \\ &\le AB\, \epsilon (1 + \|y\|^p) < \infty, \quad t \in [0, 1]. \end{aligned}$$

Consequently, by Ascoli-Arzelà compactness criteria, the map $F : \mathcal{P} \to E$ is completely continuous. Then $F : \mathcal{P} \to E$ is a 0-set contraction.

3. For $y \in E$, we have

$$(I-T)^{-1}y = \frac{y+2z_0}{\epsilon}\cdot$$

Hence,

$$\alpha\left((I-T)^{-1}z_0\right) = \alpha\left(\frac{3z_0}{\epsilon}\right) = \frac{3z_0}{\epsilon} + z_0 \geq c.$$

and

$$\beta\left((I-T)^{-1}0\right) = \beta\left(\frac{2z_0}{\epsilon}\right) = \frac{2z_0}{\epsilon} < b.$$

Suppose that $y \in \mathcal{P}$ with $\beta(y) = b$. Then

$$\alpha(Ty) = \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} Ty(t) + z_0 \geq 3z_0 > a.$$

Consequently, $(\mathcal{A}1)$ holds.

4. Let $y \in \mathcal{P}$ with $\beta(y) = b$ and $[\alpha(y) \geq a$ or $\alpha(Ty+Fy) < a]$. Then, using (4.3.3) and (4.3.4), we obtain

$$\beta(Ty) < b \quad \text{and} \quad \beta(Ty+Fy) < b.$$

Consequently, $(\mathcal{A}2)$ holds.

5. Let $y \in \mathcal{P}$ with $\alpha(y) = c$, we get

$$\|y\| \leq \frac{1}{\Lambda} \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) \leq \frac{1}{\Lambda}\alpha(y) = \frac{c}{\Lambda}.$$

Hence,

$$\beta(Ty+z_0) \leq (1-\epsilon)\frac{c}{\Lambda} + 3z_0 \leq d.$$

Consequently, $(\mathcal{A}3)$ holds.

6. Suppose that $y \in \mathcal{P}$ with $\alpha(y) = c$. Then

$$\begin{aligned}
\alpha(Ty + Fy) &= \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} (Ty(t) + Fy(t)) + z_0 \\
&= \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} \left((1-\epsilon)y(t) + \epsilon \int_0^1 G(t,s) f(s, y(s)) ds \right) \\
&\geq (1-\epsilon) \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} y(t) + \epsilon \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} \int_{\frac{\eta}{3}}^{\frac{\eta}{2}} G(t,s) f(s, y(s)) ds \\
&\geq (1-\epsilon)(c - z_0) + \epsilon \frac{\eta}{3} \left(1 - \frac{\eta}{2} \right) \tilde{A} \\
&> c.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\alpha(Ty + z_0) &= \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} (Ty(t) + z_0) + z_0 \\
&\geq \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} Ty(t) + 2z_0 \\
&\geq (1-\epsilon) \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} y(t) + 4z_0 \\
&\geq 4z_0 > c.
\end{aligned}$$

Consequently, $(\mathcal{A}4)$ holds.

7. Let $b_1 = 2\, z_0$. Then

$$\alpha(b_1) = 3\, z_0 > c \;\text{ and }\; \beta(b_1) = 2\, z_0 < b.$$

Therefore

$$\{y \in \mathcal{P} : c < \alpha(y) \quad \text{and} \quad \beta(y) < b\} \cap \Omega \neq \emptyset.$$

8. Let $y \in \mathcal{P}(\alpha, c)$. Then $y \in \mathcal{P}$ and $\alpha(y) \leq c$. Hence,

$$\|y\| \leq \frac{1}{\Lambda} \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} y(t) \leq \frac{1}{\Lambda} (c - z_0) \leq b.$$

Thus, $y \in \mathcal{P}(\beta, b)$ so $\mathcal{P}(\alpha, c) \subset \mathcal{P}(\beta, b)$ and $\mathcal{P}(\beta, b)$ is bounded. Since $0 \in \mathcal{P}(\alpha, c)$, we get $\mathcal{P}(\alpha, c) \cap \Omega \neq \emptyset$.

9. Let $\lambda \in [0,1]$ be fixed and $u \in \mathcal{P}(\beta, b)$ be arbitrary chosen. Take

$$v(t) = \frac{2(1-\lambda)z_0 + \lambda\epsilon \int_0^1 G(t,s)f(s,u(s))ds}{\epsilon}, \quad t \in [0,1].$$

We have $v(t) \geq 0$, $t \in [0,1]$, and

$$v(t) \leq \frac{\epsilon AB(1+b^p) + 2z_0}{\epsilon} \leq \rho, \quad t \in [0,1].$$

Moreover,

$$\begin{aligned}
\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} v(t) &\geq \frac{\lambda\epsilon \int_{\frac{\eta}{3}}^{\frac{\eta}{2}} \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} G(t,s)f(s,u(s))ds + 2(1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\lambda\epsilon\left(\frac{\eta}{2}-\frac{\eta}{3}\right)\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)\widetilde{A} + (1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\min\left(\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right)}{\epsilon} \\
&= \frac{\min\left(\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right)}{\epsilon\,\rho}\rho \\
&\geq \Lambda\|v\|.
\end{aligned}$$

Therefore $v \in \Omega$. Also,

$$\begin{aligned}
\lambda Fu(t) &= \epsilon\lambda\int_0^1 G(t,s)f(s,u(s))ds - \lambda 2z_0 \\
&= \epsilon\,\frac{\epsilon\int_0^1 G(t,s)f(s,u(s))ds + 2(1-\lambda)z_0}{\epsilon} - 2z_0 \\
&= \epsilon\, v(t) - 2z_0 \\
&= (I-T)v(t), \quad t \in [0,1].
\end{aligned}$$

Therefore

$$\lambda F(\mathcal{P}(\beta,b)) \subset (I-T)(\Omega).$$

10. Let $\lambda \in [0,1]$ is fixed and $\tilde{u} \in \mathcal{P}(\alpha, c)$ is arbitrarily chosen. So

$$\|\tilde{u}\| \leq \frac{1}{\Lambda}\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} \tilde{u}(t) \leq \frac{1}{\Lambda}(c - z_0) \leq b.$$

Set

$$w(t)=\frac{\lambda\epsilon\int_0^1 G(t,s)f(s,\tilde{u}(s))ds+3(1-\lambda)z_0}{\epsilon},\quad t\in[0,1].$$

We have that $w(t)\geq 0$, $t\in[0,1]$, and

$$w(t)\leq\frac{\epsilon AB(1+b^p)+3z_0}{\epsilon}\leq\rho,\quad t\in[0,1],$$

so

$$\|w\|\leq\frac{\epsilon AB(1+b^p)+3z_0}{\epsilon}\leq\rho.$$

Moreover,

$$\begin{aligned}
\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} w(t) &\geq \frac{\lambda\epsilon\int_{\frac{\eta}{3}}^{\frac{\eta}{2}}\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} G(t,s)f(s,\tilde{u}(s))ds+3(1-\lambda)z_0}{\epsilon}\\
&\geq \frac{\lambda\epsilon\left(\frac{\eta}{2}-\frac{\eta}{3}\right)\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)\widetilde{A}+(1-\lambda)z_0}{\epsilon}\\
&\geq \frac{\min\left(\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A},z_0\right)}{\epsilon}\\
&= \frac{\min\left(\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A},z_0\right)}{\epsilon\,\rho}\rho\\
&\geq \Lambda\|w\|.
\end{aligned}$$

Thus, $w\in\Omega$. Next,

$$\begin{aligned}
\lambda F\tilde{u}(t)+(1-\lambda)z_0 &= -2\lambda z_0+\lambda\epsilon\int_0^1 G(t,s)f(s,\tilde{u}(s))ds+z_0-\lambda z_0\\
&= \lambda\epsilon\int_0^1 G(t,s)f(s,\tilde{u}(s))ds+(1-3\lambda)z_0\\
&= \epsilon\frac{\lambda\epsilon\int_0^1 G(t,s)f(s,\tilde{u}(s))ds+3(1-\lambda)z_0}{\epsilon}-2z_0\\
&= \epsilon w(t)-2z_0\\
&= (I-T)w(t),\quad t\in[0,1].
\end{aligned}$$

Therefore

$$\lambda F(\mathcal{P}(\alpha,a))+(1-\lambda)z_0\subset(I-T)(\Omega).$$

By Theorem 4.2.1, it follows that the BVP (4.3.1) has at least one solution $y \in \Omega$ such that

$$\beta(y) < b \quad \text{and} \quad \alpha(y) > c.$$

□

4.3.2 An Example

Consider the BVP

$$\begin{aligned} y'' + \frac{y^2}{(200+t^2)(1+y)} + \tfrac{1}{500}(1+t) &= 0, \quad t \in (0,1), \\ y(0) = y\left(\tfrac{1}{2}\right), \quad y(1) &= 0. \end{aligned} \tag{4.3.5}$$

Here

$$f(t,y) = \frac{y^2}{(200+t^2)(1+y)} + \frac{1}{500}(1+t), \quad t \in [0,1], \quad y \in [0,\infty), \quad k=1, \quad \eta = \frac{1}{2}.$$

We have, $f \in \mathcal{C}([0,1] \times \mathbb{R}^+)$ and $0 < \dfrac{1}{500} \le f(t,y) \le a_1(t) + a_2(t)\,|y|^2$ for $t \in [0,1]$ and $y \in [0,\infty)$, where $p = 2,\ a_1(t) = \dfrac{1}{500}(1+t),\ a_2(t) = \dfrac{1}{200+t^2},\ 0 \le a_1, a_2 \le \dfrac{1}{200}$ on $[0,1]$. So, the condition $(\mathcal{C}1)$ holds.

Take the constants

$$\begin{aligned} &\epsilon = \tfrac{1}{2}, \quad B = 3, \quad A = \tfrac{1}{200}, \quad \widetilde{A} = \tfrac{1}{500}, \quad b = \tfrac{41}{50}, \quad d = \tfrac{4}{5},\ \rho = \tfrac{4}{3} \\ &c = z_0 = 2 \times 10^{-6},\ a = \tfrac{5}{2} \times 10^{-6},\ \Lambda = \frac{\min((\tfrac{1}{2}\tfrac{1}{72}\tfrac{3}{4}) \times \tfrac{1}{500},\ 10^{-6})}{\tfrac{2}{3}} = \tfrac{3}{2} \times 10^{-6} < 1. \end{aligned}$$

We have

$$z_0 \le c < \min\left(a,\ (1-\epsilon)(c-z_0) + \epsilon\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)\tilde{A}, 3z_0\right) = \frac{5}{2} \times 10^{-6},$$

$$\frac{2z_0}{\epsilon} = 4\,z_0 = 8 \times 10^{-6} < b, \quad \frac{1}{\Lambda}(c-z_0) = 0 \le b,$$

$$(1-\epsilon)\frac{c}{\Lambda} + 3z_0 = \frac{2}{3} + 6 \times 10^{-6} < d,$$

$$\frac{\epsilon AB(1+b^p)+3z_0}{\epsilon} = \frac{3}{500}\left(1+\left(\frac{41}{50}\right)^2\right) + 12\times 10^{-6} \leq \frac{4}{5} = \rho,$$
$$AB(1+b^p) = \frac{3}{200}\left(1+\left(\frac{41}{50}\right)^2\right) = \frac{25}{1000} < b.$$

Thus, $(\mathcal{C}2)$ holds. By Theorem 4.3.2, it follows that the BVP (4.3.5) has at least one nonnegative solution.

4.4 Concluding remarks

In this work, the functional Expansion-Compression fixed point theorem of Leggett-Williams type developed in [3] is extended to the class of mappings of the form $T+F$, where $(I-T)$ is Lipschitz invertible and F is a k-set contraction. As application of some obtained theoretical results, a new result on the existence of nonnegative solutions for a second order differential equation subjected to three-point boundary value problem is developed. The fixed point theorems presented in this chapter can be used to study other classes of BVPs as well as some IVPs for ODEs. For these purposes, we must first find an integral representation of the solutions of the considered IVPs/BVPs and use it to define the operators F and T.

General conclusion

This work is a contribution to fixed point theory on cones of Banach spaces for the sum of two operators and to the study of the existence of solutions for boundary value problems subjected to ordinary differential equations. More precisely, the purpose of this thesis is twofold, firstly, we develop a new fixed point theorem in cones of functional type for the class of k-set contraction perturbed by a mapping T such that $(I-T)$ is Lipschitz invertible. Secondly, we use some recent fixed point results to investigate the existence, nonnegativity, localization and multiplicity of solutions for two-point BVPs of first order as well as for three-point BVPs of second order. The study of these types of problems is driven not only by a theoretical interest, but also by the fact that several phenomena in engineering, physics, and in the life sciences can be modeled in this way. Overall, this work is a contribution to both theoretical and applied parts of the fixed point theory.

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