République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
Université A.MIRA-BEJAIA

Faculté des Sciences Exactes
Département de Mathématiques

THÈSE

Présentée par

IKASSOULENE Abderrahmane

Pour l'obtention du grade de

DOCTEUR EN SCIENCES

Filière : Mathématiques

Option : Analyse et Probabilités

Thème

Sommes d'opérateurs linéaires et application à la résolution de quelques problèmes paraboliques

Soutenue le : 03.07.2023 Devant le Jury composé de :

Nom et Prénom	Grade		
Mr BOUKOUCHA Rachid	MCA	Univ. de Bejaia	Président
Mr KHELOUFI Arezki	Professeur	Univ. de Bejaia	Rapporteur
Mr RAHMOUNE Azedine	Professeur	Univ. Bordj Bou Arréridj	Examinateur
Mr MEROUANI Abdelbaki	Professeur	Univ. de Sétif	Examinateur

Année Universitaire : 2022/2023

Aknowledgments

First of all, I would like to express my sincere gratitude to my supervisor Prof. **KHELOUFI Arezki** for having proposed this subject and agreed to supervise this thesis. I thank him for his patience, his availability and his valuable advice during the preparation of this work.

I sincerely thank the chairman of the jury committee, Dr. **BOUKOUCHA Rachid**, and the jury members, Prof. **RAHMOUNE Azedine** and Prof. **MEROUANI Abdelbaki** for the attention they paid to this thesis and for their participation in the jury.

Finally, I express my deepest gratitude to all my friends and family for their support throughout my years of study.

Abstract

In this work we study a linear $2m$-th order parabolic equation, subject to Dirichlet type condition on the lateral boundary, where the right-hand side of the equation is taken in the Lebesgue space L^p, defined on a time-varying domain of $\mathbb{R}^{N+1}$. The approach is based on the use of the operators' sum method in Banach spaces; we use Labbas-Terreni results on the operators' sum theory in the non-commutative case. We are especially interested in the question of what sufficient conditions, as weak as possible, the dimension N, the exponent p and the type of the domain must verify in order that our problem has a solution with optimal regularity.

This thesis is divided into three chapters:

In the first chapter we recall some basic tools and concepts of elementary functional analysis which are necessary in the operators' sum theory.

In the second chapter we will refer to the essential of the operators' sum method, that we will use in the chapter three.

The third chapter is devoted to present our results.

Keywords: High order parabolic equation, non-cylindrical domains, anisotropic Sobolev spaces, sum of linear operators, interpolation spaces.

Résumé

Dans ce travail, nous étudions une équation parabolique linéaire d'ordre $2m$, sous une condition de type Dirichlet sur la frontière latérale. Le membre droit de l'équation est pris dans l'espace de Lebesgue L^p. L'équation est définie dans un domaine de $\mathbb{R}^{N+1}$. L'approche est basée sur l'utilisation de la méthode des somme d'opérateurs dans les espaces de Banach; nous utilisons les résultats de Labbas-Terreni de la théorie des somme d'opérateurs dans le cas non commutatif. Nous nous intéressons plus particulièrement à la question de savoir quelles conditions suffisantes, aussi faibles que possible, la dimension N, l'exposant p et le type du domaine doivent vérifier pour que notre problème ait une solution avec une régularité optimale.

Cette thèse est divisée en trois chapitres :

Dans le premier chapitre nous rappelons quelques outils et concepts de base de l'analyse fonctionnelle élémentaire dont nous aurons besoin dans la théorie des somme d'opérateurs.

Dans le deuxième chapitre nous ferons référence à l'essentiel de la méthode des somme d'opérateurs, que nous utiliserons dans le chapitre trois.

Le troisième chapitre est consacré à la présentation des résultats obtenus dans l'article.

Mots clés: Équation parabolique d'ordre supérieur, domaines non cylindriques, espaces de Sobolev anisotropes, somme d'opérateurs linéaires, espaces d'interpolation.

ملخص

في هذَا العَمل، ندرس معَادلة خطية مكَافئِة من درجة زوجية كيفية، تحت شرط من نوع ديريكلي علَى الحد الجَانبي. الطرف الأَيمن من المعَادلة يُؤخذ في فضَاء لوبيغ L^p . المعَادلة معرفة علَى ميدَان من $\mathbb{R}^{n+1}$. اَلمقَاربة تعتمد علَى استخدَام طريقة **مجموع** المُؤثرَات في فضَاءَات بنَاخ، نستخدم نتَائِج Labbas-Terreni في نظرية **مجموع** المُؤثرَات في الحَالة غير التبديلية. نحن مهتمون بشكل خَاص بتحديد الشروط الكَافية، اَلضعيفة قدر الاَمكَان، علَى البعد N و الأَس p و نوع المجَال، اَلتي يجب استيفَاُؤهَا حتَى يكون لَلمسأَلة حل مع انتظَام مثَالي.

تنقسم هذه الرسَالة إلَى ثلَاثة فصول :

في الفصل الأَوَل نستذكر بعض الأَدوَات و المفَاهيم الأَسَاسية لِلتحليل الدَالي، اَلتي سنحتَاجهَا في نظرية **مجموع** المُؤثرَات.

في الفصل الثَاني سوف نعرض الجزء الأَهم من نظرية **مجموع** المُؤثرَات، و التي سنستخدمهَا في الفصل الثَالث.

اَلفصل الثَالث مخصَص لعرض النتَائِج التي تم الحصول عليهَا في المقَال.

اَلكلمَات المفتَاحية : معَادلة مكَافئِة عَالية الدرجة. اَلميدَان غير الأَسطوَاني. فضَاءَات سوبولَاف. **مجموع** المُؤثرَات الخطية. فضَاءَات الاَستقطَاب.

Publication

Abderrahmane Ikassoulene and Arezki Kheloufi, L^p-regularity results for 2m-th order parabolic equations in time-varying domains, *Miskolc Mathematical Notes*, Vol. 23 (2022), No. 1, pp. 211-230, DOI: 10.18514/MMN.2022.3192

Contents

General introduction

In science in general and in mathematics in particular one often encounters equations of the form

$$Au + Bu = f, \tag{1}$$

where f is a given element of a vector space X, A and B are two closed (unbounded) linear operators in X with domains $D(A)$ and $D(B)$ respectively, and $u \in D(A) \cap D(B)$ is the unknown (or the solution) to be determined. It is clear that if we have no or little knowledge of the operators A and B, then little can be said about the existence and regularity of solutions to the equation. Among the important general theories developed for equations of the form (1), we can cite the operators' sum method.

The operators' sum method is developed by G. Da Prato and P. Grisvard in 1975 (see [7]), then by G. Dore and A. Venni in 1987 (see [9]). It allows us to give a unified treatment to problems to all appearances completely different in nature, like Cauchy and Dirichlet problems. This method gives spectral properties of the sum operator $L = A+B$ from those of the linear operators A and B. It gives conditions under which the abstract equation (1) can be solved. The original idea of Grisvard refers to parabolic and elliptic operators. So, for parabolic problems, the following conditions will be imposed

$$\left\{\begin{array}{l} (A-z)^{-1} \text{ and } (B-z)^{-1} \text{ exist for } z \in \Sigma_A \text{ and } z \in \Sigma_B, \text{where} \\ \sum_A \text{ and } \Sigma_B \text{ are two sectors of the form } \{z \in \mathbb{C}/|\arg z| < \\ < \pi - \varphi\} \text{ with } \varphi = \theta_A \text{ and } \varphi = \theta_B \text{ respectively and} \\ \theta_A + \theta_B < \pi \text{ and} \\ \|(A-z)^{-1}\|_{\mathcal{L}(X)} = 0\left(\frac{1}{|z|}\right) \text{ and } \|(B-z)^{-1}\|_{\mathcal{L}(X)} = 0\left(\frac{1}{|z|}\right) \\ \text{ for } z \in \Sigma_A \text{ and } \Sigma_B \text{ respectively.} \end{array}\right. \tag{2}$$

This parabolic problem (1)-(2) can be treated by the operators' sum method in two separate cases depending on whether A and B are resolvent commuting

$$\begin{cases} \forall \lambda \in \rho(A), \forall \mu \in \rho(B) \\ (A-\lambda I)^{-1}(B-\mu I)^{-1} = (B-\mu I)^{-1}(A-\lambda I)^{-1}, \end{cases}$$

or not. In both cases the operator S_λ

$$f \in X \longmapsto S_\lambda f = -\frac{1}{2i\pi}\int_{\gamma_\lambda} (A-\lambda I - zI)^{-1}(B+zI)^{-1} f dz,$$

to be defined in the second chapter, will play a fundamental role in expressing and analyzing solutions to the equation (1). The operator S_λ allows us to give a unique and explicit solution to (1) for all f in an interpolation space $D_A(\theta, p)$ between X and $D(A)$, or in an interpolation space $D_B(\theta, p)$ between X and $D(B)$. Here we assume that $0 < \theta < 1$ and $1 \leq p \leq +\infty$. The restrictions that we impose on A and B in order to achieve this are, beside sectoriality,

$$\begin{cases} (i) \ \ \left\| (A+\lambda_0 I)\,(A+\lambda I)^{-1}\left[(A+\lambda_0 I)^{-1};(B+\mu I)^{-1}\right]\right\|_{L(E)} \\ \leq \dfrac{C}{|\lambda|^{1-\tau}\,|\mu|^{1+\rho}}, \quad \forall \lambda \in \rho\,(-A) \ \ \forall \mu \in \rho\,(-B)\,, \\ (ii) \ \ 0 \leq \tau < \rho \leq 1, \end{cases}$$

in the non-commutative case. In addition to guaranteeing the existence of a unique solution u for f in one of the above mentioned interpolation spaces, the restrictions also ensure maximal regularity of the problem with respect to the interpolation spaces in question. For example, if $f \in D_A(\theta, p)$, then not only u belongs to $D(A) \cap D(B) \subset D_A(\theta, p)$, but also Au and Bu belong to this interpolation space.

The operators' sum method may be used to investigate solutions to a number of problems related to partial differential equations. In chapter 3 we will apply it, in non commutative case, to $2m-th$ order parabolic equation:

$$\partial_t u + (-1)^m \sum_{k=1}^{m} \partial_{x_k}^{2m} u = f, \tag{3}$$

subject to Dirichlet type condition $\partial_\nu^l u = 0,\ l = 0, 1, ..., m-1$, on the lateral boundary, where m is a positive integer. The right-hand side f of the equation is taken in the Lebesgue space L^p, $1 < p < +\infty$. The problem is set in a domain of the form

$$\Omega = \{(t, x_1, x_2, ..., x_N) \in \mathbb{R}^{N+1} : 0 \leq \sqrt{x_1^2 + x_2^2 + ... + x_N^2} < t^\alpha\}, \text{ with } \alpha > 1/2m.$$

This thesis is made up of three chapters, and is organized as follows:

Chapter 1

It is devoted to the fundamental reminders as well as to the tools necessary for this work. We will first present some functional spaces such as Lebesgue spaces and Sobolev spaces. We will then make a section on some generalities on linear operators. We will present a section dealing with semi-groups, in which we will present strongly continuous semi-groups and then analytic semigroups. The last section is devoted to interpolation spaces and their properties.

Chapter 2

This chapter will be devoted to the operators' sum theory, especially in the parabolic case. It is composed of two sections. In the first, we will expose the theory of sums in commutative case, and in the last section we will deal with the non-commutative case.

Chapter 3

In this last chapter we will solve our problem in three Steps:

In Step1 we perform a change of variables conserving (modulo a weight) the spaces L^p and $H_p^{1,2m}$, and transforming Problem (3) into a degenerate parabolic problem in a cylindrical domain.

Step 2 is concerned with the application of the sum of operators' method to the transformed problem. We can find in the Favini-Yagi book [10] an important study of abstract problems of parabolic type with degenerated terms in the time derivative. They used the notion of multi-valued linear operators and constructed fundamental solutions when the right-hand side has a Hölder regularity with respect to the time. Our approach is based on the direct use of operators' sums in a weighted L^p-Sobolev space.

Finally, in Step 3 we give results concerning the transformed problem and we return to our initial problem by using an inverse change of variables.

The thesis ends with a conclusion and prospects.

Chapter 1

Preliminaries

In this chapter, we recall some functional spaces, and some definitions and results on linear operators, as well as semigroups and interpolation spaces.

1.1 Some functional spaces

1.1.1 L^p spaces

Let Ω be an open set of $\mathbb{R}^n$, with $n \in \mathbb{N}^*$.

Definition 1.1.1. Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we set

$$L^p(\Omega) = \Big\{ u : \Omega \to \mathbb{R},\ u \text{ is measurable and } \int_\Omega |u(x)|^p dx < +\infty \Big\},$$

with norm

$$\|u\|_{L^p(\Omega)} = \left[\int_\Omega |u(x)|^p dx \right]^{\frac{1}{p}}.$$

We set

$$L^\infty(\Omega) = \Big\{ u : \Omega \to \mathbb{R},\ u \text{ is measurable and } \ |u(x)| \leq C \text{ a.e. in } \Omega \text{ for some canstant } C \Big\},$$

with the norm

$$\|u\|_{L^\infty(\Omega)} = \inf \Big\{ C,\ |u(x)| \leq C \text{ a.e. on } \Omega \Big\}.$$

We have the following properties:

1. L^p is a Banach space for any p, $1 \leq p \leq \infty$.

2. The dual of L^p is L^q, for any p, $1 < p < +\infty$, where q is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. The dual of L^1 is L^∞. The dual of L^∞ is strictly bigger than L^1.

3. Hölder's inequality:
   Assume that $u \in L^p$ and $v \in L^q$ with $1 \leq p, q \leq +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $uv \in L^1$ and
   $$\int_\Omega \big|u(x)v(x)\big|dx \leq \|u\|_{L^p}\|v\|_{L^q}.$$

4. L^2 equipped with the scalar product
   $$(u, v) = \int_\Omega u(x)v(x)dx,$$
   is the unique Hilbert space among all L^p spaces.

5. If $u \in L^\infty$ then we have
   $$|u(x)| \leq \|u\|_{L^\infty} \text{ a.e. on } \Omega.$$

1.1.2 Sobolev spaces

Let Ω be an open set of $\mathbb{R}^n$, and let $p \in \mathbb{R}$ with $1 \leq p \leq +\infty$.

The space $W^{m,p}(\Omega)$

Let $m \geq 0$ an integer.

Definition 1.1.2. The Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \Big\{u \in L^p(\Omega) \,/ \partial^\alpha u \in L^p(\Omega),\ \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m\Big\},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We set $H^m(\Omega) = W^{m,2}(\Omega)$.

The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \left(\sum_{|\alpha|=0}^{m} \|\partial^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty,$$

and

$$\|u\|_{W^{m,\infty}} = \max_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^\infty}.$$

The space $H^m(\Omega)$ is equipped with the scalar product

$$(u,v)_{H^m} = \sum_{|\alpha|=0}^{m} (\partial^\alpha u, \partial^\alpha v)_{L^2} = \sum_{|\alpha|=0}^{m} \int_\Omega \partial^\alpha u . \partial^\alpha v \; dx,$$

and with the associated norm

$$\|u\|_{H^m} = \left(\sum_{|\alpha|=0}^{m} \|\partial^\alpha u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

It is easy to show that

$$\cdots \subset W^{3,p} \subset W^{2,p} \subset W^{1,p} \subset W^{0,p} = L^p.$$

Hereafter some properties of the Sobolev space $W^{m,p}(\Omega)$:

1. The space $W^{m,p}$ is a Banach space for $1 \leq p \leq +\infty$.

2. H^m is the unique Hilbert space among all $W^{m,p}$ spaces.

3. Sobolev inequality (Sobolev embedding): There exists a constant C such that

$$\|u\|_{W^{m,\infty}(\Omega)} \leq C\|u\|_{W^{m,p}(\Omega)}, \;\; \forall u \in W^{m,p}(\Omega), \;\; \forall \; 1 \leq p \leq +\infty.$$

In other words, $W^{m,p}(\Omega) \subset W^{m,\infty}(\Omega)$ with continuous injection for all $1 \leq p \leq +\infty$. We also write $W^{m,p}(\Omega) \hookrightarrow W^{m,\infty}(\Omega)$.

Remark 1.1.1. **(Continuous embedding / Compact embedding)**
Let X and Y be two normed vector spaces, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively, such that $X \subseteq Y$.

- ▶ We say that X is continuously embedded in Y if the identity function

$$\begin{aligned} i : X &\longrightarrow Y, \\ x &\longmapsto x, \end{aligned}$$

is continuous, i.e. if there exists a constant $C \geq 0$ such that

$$\|x\|_Y \leq C\|x\|_X, \;\; \forall x \in X.$$

► We say that X is compactly embedded in Y if

- X is continuously embedded in Y.
- The identity function i of X into Y is a compact operator, i.e., any bounded subset in X is relatively compact subset in Y (or in other words for any bounded sequence (x_n) in X, there exists a subsequence (x_{n_k}) that converges in Y).

The space $W_0^{m,p}(\Omega)$

Let $1 \leq p < +\infty$, and let $m \geq 2$ be an integer.

Definition 1.1.3. $W_0^{m,p}(\Omega)$ is defined as the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$, i.e.,

$$\overline{\mathcal{C}_c^\infty(\Omega)}^{W^{m,p}(\Omega)} = W_0^{m,p}(\Omega) \subseteq W^{m,p}(\Omega).$$

We set $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

Remark 1.1.2. We can also define $W_0^{m,p}(\Omega)$ as follow

$$W_0^{m,p}(\Omega) = \Big\{ u \in W^{m,p}(\Omega) \ /\partial^\alpha u = 0 \text{ on } \partial\Omega, \ |\alpha| < m \Big\}.$$

Hereafter some properties of the spaces $W_0^{m,p}(\Omega)$:

1. The space $W_0^{m,p}$ is a Banach space for $1 \leq p < +\infty$.
2. H_0^m is a Hilbert space.

1.2 Linear operators

Let E and F be Banach spaces over the same (real or complex) field $\mathbb{K}$ with the norms denoted by $\|\cdot\|_E$ and $\|\cdot\|_F$ or just by $\|\cdot\|$.

1.2.1 Generalities

Definition 1.2.1. A mapping $T : D(A) \subset E \to F$ is said to be a linear operator if $T(ax + by) = aT(x) + bT(y)$ for all x, y in $D(A)$ and for all a, b in $\mathbb{K}$. If Y = $\mathbb{K}$ then T is called a linear form. If $T : D(A) \subset E \to F$ is not a linear operator, then T is often referred to as a nonlinear operator (or just a mapping).

1. $D(A)$ or just D_A is said to be the domain of A. It is a vector subspace of E, and for all $x \in D(A)$ we can denote $A(x)$ by Ax.

2. We use the notation $(A, D(A))$, to denote the operator A with domain $D(A)$.

3. The set of linear operators from E to F is denoted by $L(E, F)$.

4. When $F = E$, A is said to be a linear operator in E, and we write $A \in L(E)$ or $(A, D(A)) \in L(E)$.

Definition 1.2.2. Let $A : D(A) \subset E \to F$ be a linear operator.

1. We define the graph of A by

$$\mathcal{G}(A) := \{(x, y) \in E \times F / x \in D(A), y = Ax\} = \{(x, Ax) \in E \times F / x \in D(A)\},$$

which is a vector subspace (we also say a linear subspace) of $E \times F$.

2. We define the image of A as a linear subspace of F, by

$$Im(A) := \{y \in F / \exists x \in D(A), y = Ax\} = \{Ax \in F / x \in D(A)\},$$

which will be referred to as the range of the operator A.

3. We define the kernel of A by

$$\ker(A) := \{x \in D(A) / Ax = 0\},$$

wich is a vector subspace of E.

4. We say that A has a dense domain if $\overline{D(A)} = E$.

Definition 1.2.3. Let A and B be two linear operators in E. The operator AB is defined by

$$\begin{cases} D(AB) = \{x \in D(B) : Bx \in D(A)\}, \\ (AB)x = A(Bx) \ \ \forall x \in D(B). \end{cases}$$

We define, $A^n,\ n \in \mathbb{N}$, by

$$\begin{cases} D(A^0) = E \text{ and } A^0 = I, \\ D(A^1) = D(A) \text{ and } A^1 = A, \\ \forall n \geq 2\ \ D(A^n) = \{x \in D(A^{n-1}) : A^{n-1}x \in D(A)\} \text{ and } A^n = AA^{n-1}. \end{cases}$$

Definition 1.2.4. Let A and B be two linear operators from E to F. We say that B is an extension of A and we denote $A \subset B$ if

1. $D(A) \subset D(B)$,

2. $\forall x \in D(A),\ Ax = Bx$.

Conversely we say that A is a restriction of B, and we write $B_{|D(A)} = A$.

Proposition 1.2.1. *Suppose that $A, B \in L(X, Y)$ satisfy: $A \subset B, KerB = 0$, and $ImA = Y$. Then $A = B$.*

Definition 1.2.5. Let A be a linear operator on E. If A is injective, we define the operator A^{-1} by

$$\begin{array}{llll} A^{-1}: & Im(A) & \longrightarrow & E \\ & y & \longmapsto & A^{-1}y = x \end{array}$$

where $x \in D(A)$ is defined by $Ax = y$. Note that $Im(A^{-1}) = D(A)$.

1.2.2 Bounded linear operators

Definition 1.2.6. Let A be a linear operator from E to F. We say that A is bounded if $\sup\{\|Ax\|; x \in D(A), \|x\| \leq 1\} < +\infty$. Otherwise, the operator $(A, D(A))$ is said to be unbounded.

The space of bounded linear operators from E to F is noted by $\mathcal{L}(E, F)$. If $E = F$, we pose $\mathcal{L}(E) := \mathcal{L}(E, E)$.

Proposition 1.2.2. *Let A be a linear operator from E to F. Then, the following properties are equivalent:*

1. *$A \in \mathcal{L}(E, F)$,*

2. $\forall x_0 \in E, \lim_{x \to x_0} \|Ax - Ax_0\|_F = 0,$

3. $\lim_{x \to 0} \|Ax\|_F = 0,$

4. $\exists M \geq 0, \ \forall x \in E, \ \|Ax\|_F \leq M\|x\|_E.$

Definition 1.2.7. Let $A \in \mathcal{L}(E, F)$. We define the norm of A as follows:

$$\|A\|_{\mathcal{L}(E,F)} = \inf\{c > 0, \|Ax\|_F \leq c\|x\|_E, \forall x \in E\}.$$

Remark 1.2.1.

1. The space $\mathcal{L}(E, F)$ equipped with the norm $\|.\|_{\mathcal{L}(E,F)}$ is a Banach space.

2. An operator is bounded if and only if its norm is finite.

Definition 1.2.8. Two operators $A, B \in \mathcal{L}(E)$ are said to commute if $AB = BA$

It is not easy to extend this definition to unbounded operators due to the difficulties with defining the domains of the composition. The extension is usually done to the case when one of the operators is bounded. Thus, an operator $A \in L(E)$ is said to commute with $B \in \mathcal{L}(E)$ if $BA \subset AB$. This means that for any $x \in D(A), Bx \in D(A)$ and $BAx = ABx$.

Definition 1.2.9. Let $A : E \to E$, be a linear operator. We say that A is invertible if there exists $A' \in L(E)$ such that

$$AA' = A'A = I,$$

where I is the identity operator on E. This operator A' if it exists is unique, it is called the inverse of A, and we denote it by A^{-1}.

We have the following useful conditions for invertibility of an operator.

Proposition 1.2.3. *Let E and F be Banach spaces and $A \in L(E, F)$. The following assertions are equivalent.*

1. A is invertible,

2. *$ImA = F$ and there exists $m > 0$ such that $\|Ax\| \geq m\|x\|$ for all $x \in D(A)$,*

3. *A is closed, $\overline{ImA} = F$ and there exists $m > 0$ such that $\|Ax\| \geq m\|x\|$ for all $x \in D(A)$,*

4. *A is closed, $ImA = F$, and $KerA = \{0\}$.*

Proposition 1.2.4. *Let $A \in \mathcal{L}(E)$. If $\|A\|_{\mathcal{L}(E)} < 1$, then*

1. *$(I - A)$ is invertible in $\mathcal{L}(E)$, and*

2. *$(I - A)^{-1} = \sum_{n=0}^{+\infty} A^n$.*

Definition 1.2.10. Let A and B be two linear operators on E.

1. The resolvent set of A, denoted by $\rho(A)$ or ρ_A, is defined by

$$\rho(A) := \{\lambda \in \mathbb{C} : \ (\lambda I - A)^{-1} \in \mathcal{L}(E)\}.$$

2. If $\lambda \in \rho(A)$, we define the resolvent $R(\lambda, A)$ of A at point λ by

$$R(\lambda, A) := (\lambda I - A)^{-1}.$$

3. We define the resolvent commutator as follows

$$[(A - \lambda I)^{-1}, (B - \mu I)^{-1}] = (A - \lambda I)^{-1}(B - \mu I)^{-1} - (B - \mu I)^{-1}(A - \lambda I)^{-1}.$$

4. The spectrum A, noted $\sigma(A)$ or σ_A, is defined by

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

Remark 1.2.2. In general, it is possible that either $\sigma(A)$ or $\rho(A)$ is empty. The spectrum is usually subdivided into three subsets.

1. Point spectrum $\sigma_p(A)$ is the set of $\lambda \in \sigma(A)$ for which the operator $\lambda I - A$ is not one-to-one (injective). In other words, $\sigma_p(A)$ is the set of all eigenvalues of A.

2. Continuous spectrum $\sigma_c(A)$ is the set of $\lambda \in \sigma(A)$ for which the operator $\lambda I - A$ is one-to-one and its range is dense in E but not equal to E.

3. Residual spectrum $\sigma_r(A)$ is the set of $\lambda \in \sigma(A)$ for which the operator $\lambda I - A$ is one-to-one and its range is not dense in E.

Proposition 1.2.5. *Let $A : \ D(A) \subset E \to F$ be a linear operator. Then, for all $\lambda, \ \mu \in \rho(A)$, we have*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

1.2.3 Closed linear operators

Definition 1.2.11. Let A be a linear operator from E to F. We say that the operator A is closed if his graph $\mathcal{G}(A)$ is closed in $E \times F$. Note that bounded linear operator is a closed linear operator. The space of closed linear operators from E to F is noted by $\mathcal{F}(E, F)$.

The following proposition gives an equivalent definition for a closed linear operator.

Proposition 1.2.6. *Let $A : \ D(A) \subset E \to F$ be a linear operator. A is said to be closed if and only if for any sequence $(x_n)_{n\geq 0}$ of $D(A)$ such that*

$$\begin{cases} x_n \to x, \ in \ E, \\ Ax_n \to y, \ in \ F, \end{cases}$$

one has $x \in D(A)$ and $Ax = y$.

Remark 1.2.3. For any operator A, its domain $D(A)$ is a normed space under the graph norm

$$\|x\|_{D(A)} := \|x\|_E + \|Ax\|_F.$$

The operator $A : D(A) \to F$ is always bounded with respect to the graph norm, and A is closed if and only if $D(A)$ is a Banach space under the graph norm.

Proposition 1.2.7. *Let A and B be two linear operators on E. One has*

1. *If B is bounded then B is closed.*

2. *If A is closed and B is bounded, then $A + B$ is closed.*

3. *If A is closed and B is bounded, then AB is closed.*

Theorem 1.2.1. ***(Closed graph theorem)***
Let E and F be two Banach spaces. Let A be a closed linear operator from E to F. If $D(A) = E$, then $A \in \mathcal{L}(E, F)$.

1.2.4 Compact linear operators

Definition 1.2.12. Let E and F be two Banach spaces and $T \in \mathcal{L}(E, F)$, T is said to be compact if for each sequence $(x_n)_{n\geq 1} \in E$ with $\|x_n\| = 1$ for each $n \in \mathbb{N}^\star$, the sequence $(Tx_n)_{n\geq 1}$ has a subsequence which converges in F. Equivalently, T is compact if for each bounded sequence $(x_n)_{n\geq 1} \in E$, the sequence $(Tx_n)_{n\geq 1}$ has a subsequence which converges in F.

We denote by $\mathcal{K}(E, F)$ the space of all compact operators from E to F. And if $E = F$, we write $\mathcal{K}(E)$ the space of all compact operators on E.

Theorem 1.2.2.

1. *$\mathcal{K}(E, F)$ is closed in $\mathcal{L}(E, F)$.*

2. *If $T \in \mathcal{K}(E, F)$, $R \in \mathcal{L}(G, E)$ and $S \in \mathcal{L}(F, G)$, then $S \circ T \in \mathcal{K}(E, G)$ and $T \circ R \in \mathcal{K}(G, F)$.*

Theorem 1.2.3. *Let $T \in \mathcal{K}(E)$, then*

1. *$dim Ker(I - T)$ is finite.*

2. *$Im(I - T)$ is closed.*

3. *$Ker(I - T) = \{0\} \Leftrightarrow Im(I - T) = E$.*

1.2.5 Sectorial operators

It is important to clarify that there are many equivalent definitions for sectorial operators. Here we will use the following definition:

Definition 1.2.13. Let E be a complex Banach space and A be a closed linear operator in E. Then, A is said to be sectorial if

(i) $D(A)$ and $Im(A)$ are dense in E,

(ii) $Ker(A)=\{0\}$,

(iii) $]-\infty,0[\subset\rho(A)$ ($\rho(A)$ is the resolvent set of A) and there exists a constant $K\geq 1$ such that $\forall t>0$, $\left\|t(A+tI)^{-1}\right\|_{L(E)}\leq K$.

Remark 1.2.4. We will see later that if A is sectorial, then $\rho(-A)$ contains an open sector $\sum_{\varphi}=\{z\in\mathbb{C}:\ z\neq 0,\ \ |\arg z|<\varphi\}$, with $\varphi\in\,]0,\pi[$.

We can also find the following definition:

Definition 1.2.14. Let $0<\omega\leq\frac{\pi}{2}$. We define the following sector

$$\Sigma_{\omega}=\{z\in\mathbb{C}/\{0\}:|\arg z|<\omega\}.$$

1) A closed linear operator A on E is said to be sectorial of angle ω if

$$\sigma_A\subset\overline{\Sigma_{\omega}},$$

and

$$\forall\omega'\in]\omega,\pi[,\ \sup_{\lambda\notin\overline{\Sigma_{\omega'}}}||\lambda(A-\lambda I)^{-1}||<+\infty.$$

2) We denote by $Sect(\omega)$ the set of linear operators on E which are sectorials of angle ω.

1.2.6 Some theorems of functional analysis

Banach-Steinhaus theorem

Theorem 1.2.4.

Let E and F be two Banach spaces. Let $(A_i)_{i\in I}$ be a family (not necessarily countable) of *continuous linear operators from E to F. We suppose that*

$$\sup_{i\in I} \|A_i x\| < +\infty, \ \forall x \in E.$$

Then

$$\sup_{i\in I} \|A_i\| < +\infty.$$

Fubini's theorem

Let M be a metric space, we denote by $\mathcal{B}(M)$ the Borel σ-algebra on M, that is the collection that contains all open and closed sets, all countable unions and intersections of closed or open sets, and so on. We write $\mathcal{B}_d$ instead of $\mathcal{B}(\mathbb{R}^d)$.

Let $A \in \mathcal{B}_n$, $B \in \mathcal{B}_m$, and $(x, y) \in A \times B \subseteq \mathbb{R}^{n+m}$. It can be seen that $A \times B \in \mathcal{B}_{n\times m}$.

Let $f : A \times B \to [0, +\infty]$. We set

$$f_y : A \to [0, +\infty[, \ \ f_y(x) = f(x, y) \text{ for each fixed } y \in B,$$

$$f_x : B \to [0, +\infty[, \ \ f_x(y) = f(x, y) \text{ for each fixed } x \in A.$$

These functions are measurable (for each fixed $y \in B$, resp. $x \in A$). Now we can state Fubini's theorem, which we will use later.

Theorem 1.2.5.

a) Let $f : A \times B \to [0, +\infty]$ be a measurable function. Then the functions $F : A \to [0, \infty]$ and $G : B \to [0, \infty]$, given by

$$F(x) = \int_B f_x(y)dy \ \ \text{for } x \in A, \quad G(y) = \int_A f_y(x)dx \ \ \text{for } y \in B,$$

are measurable, and it holds

$$\int_{A\times B} f(x, y)d(x, y) = \int_A (\int_B f(x, y)dy)dx = \int_B (\int_A f(x, y)dx)dy. \tag{*}$$

b) Let $f \in L^1(A \times B)$. Then there are null sets $N_A \subseteq A$ and $N_B \subseteq B$ such that f_x is integrable for all $x \in A\backslash N_A$ and f_y is integrable for all $y \in B\backslash N_B$. We define F and G as above for $x \in A\backslash N_A$ and for $y \in B\backslash N_B$, respectively, and we put $F(x) = 0$ and $G(y) = 0$ for $x \in N_A$ and $y \in N_B$, respectively. Then F and G are integrable and formula (*) holds.

Cauchy's Theorem

Let U be an open set of $\mathbb{C}$. We denote by $H(U)$, the space of holomorphic functions, from U in $\mathbb{C}$. Let $g \in H(U)$, K a compact set with boundary in U and z_0 inside K, then

$$g(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{g(z)}{z - z_0} dz,$$

where γ is the positively oriented boundary of K.

Dunford Integral

Let E be a complex Banach space and A be a closed linear operator. We denote by $H(A)$ the space of variable complex functions which are holomorphic in a closed set containing the spectrum of A. The formula analogous to the Cauchy formula for the holomorphic functions is defined by the following Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_\gamma f(z) \, (z - A)^{-1} \, dz,$$

where γ is a simple curve and $f \in H(A)$. The operator $f(A) \in \mathcal{L}(E)$ and does not depend on γ.

1.3 Semigroups of linear operators

1.3.1 Strongly continuous semigroup

Let $(X, \|.\|)$ be a complex Banach space.

Definitions and Propreties

Definition 1.3.1. The family of bounded linear operators $(T(t))_{t\geq 0}$ in X is said to be a semigroup if it verifies

1. $T(0) = I$,

2. $T(t_1 + t_2) = T(t_1)T(t_2), \ \ \forall t_1, \ t_2 \geq 0.$

Definition 1.3.2. A semigroup $(T(t))_{t\geq 0}$ is said to be strongly continuous if, for all $f \in X$,

$$\lim_{t\to 0^+} \|T(t)f - f\| = 0.$$

A strongly continuous semigroup, is called a $\mathcal{C}_0$-semigroup.

Example 1.3.1. *We consider the space*

$$E := \{f : [0, +\infty[\to \mathbb{R} : \ f \textit{ is uniformly continuous and bounded}\}.$$

equipped with the norm

$$\|f\| = \sup_{x\in[0,+\infty[} |f(x)|,$$

E becomes a Banach space. Let $(T(t))_{t\geq 0}$ be the family of operators defined on E by

$$(T(t)f)(x) = f(t + x), \ \ \forall t \geq 0, \ \forall f \in E, \ \forall x \in [0, +\infty[.$$

$(T(t))_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup of bounded linear operators on E, called $\mathcal{C}_0$-semigroup of right translation.

Proposition 1.3.1. *If $(T(t))_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup on X, then there are constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\forall t \geq 0, \ \|T(t)\| \leq Me^{\omega t}.$$

Remark 1.3.1.

1. If $\omega = 0$, the $\mathcal{C}_0$-semigroup $(T(t))_{t\geq 0}$ is called uniformly bounded. In this case, we have

$$\|T(t)\| \leq M.$$

2. If $\omega = 0$ and $M = 1$, $(T(t))_{t\geq 0}$ is called $\mathcal{C}_0$-semigroup of contraction. In this case, we have

$$\|T(t)\| \leq 1.$$

3. If $M = 1$ and $\omega \in \mathbb{R}$, the $\mathcal{C}_0$-semigroup $(T(t))_{t\geq 0}$ is called quasi-continuous. In this case, we have

$$\|T(t)\| \leq e^{\omega t}.$$

The following corollary is a direct consequence of the above proposition.

Corollary 1.3.1. *Let $(T(t))_{t\geq 0}$ be a $\mathcal{C}_0$-semigroup on X. Then for all $x \in X$, the function $t \mapsto T(t)x$ is continuous from $\mathbb{R}^+$ to X.*

1.3.2 Generators

We have already defined the $\mathcal{C}_0$-semigroups, now we are going to associate with them a very important element that is the (infinitesimal) generator whose definition is as follows:

Definition 1.3.3. Let $(T(t))_{t\geq 0}$ be a $\mathcal{C}_0$-semigroup on X. We call an infinitesimal generator (or just generator) of $(T(t))_{t\geq 0}$, the operator A defined on the set

$$D(A) = \{f \in X : \lim_{t\to 0^+} \frac{T(t)f - f}{t} \ \ exists \ \ in \ \ X\},$$

by

$$Af = \lim_{t\to 0^+} \frac{T(t)f - f}{t}, \ \forall f \in D(A).$$

Example 1.3.2. *We mentioned above that the family of operators $(T(t))_{t\geq 0}$ defined on*

$$E := \{f : [0, +\infty[\to \mathbb{R} : \ f \ \textit{is uniformly continuous and bounded}\}$$

by

$$(T(t)f)(x) = f(t + x), \ \ \forall t \geq 0, \ \forall f \in E, \ \forall x \in [0, +\infty[,$$

is a $\mathcal{C}_0$-semigroup. Then we can show that its generator is the operator A defined by

$$D(A) = \{f \in E : \ f' \in E\} = \ \ and \ \ Af = f'.$$

Proposition 1.3.2. *Let $(T(t))_{t\geq 0}$ be a $\mathcal{C}_0$-semigroup on X and A be its generator, then*

1.

$$\lim_{h\to 0^+} \frac{1}{h} \int_t^{t+h} T(s)x \ ds = T(t)x, \ \forall t \geq 0, \ \forall x \in X.$$

2. $\forall t \geq 0,\ \forall x \in X,\ \int_0^t T(s)x\ ds\ \in D(A)$ *and we have*

$$A(\int_0^t T(s)x\ ds) = T(t)x - x.$$

3. $\forall t \geq 0,\ \forall x \in D(A),\ T(t)x \in D(A)$ *and we have*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

4. $\forall t \geq 0,\ \forall s \geq 0,\ \forall x \in D(A),$ *we have*

$$T(t)x - T(s)x = \int_s^t AT(\tau)xd\tau = \int_s^t T(\tau)Axd\tau.$$

Corollary 1.3.2. *Let $(T(t))_{t\geq 0}$ be a $\mathcal{C}_0$-semigroup on X and A be its generator. Then A is closed and its domain is dense in X.*

Definition 1.3.4. A $\mathcal{C}_0$-semigroup $(T(t))_{t\geq 0}$ of generator A can have an extension to a group $(U(t))_{t\in\mathbb{R}}$, if and only if, $(-A)$ generates a $\mathcal{C}_0$-semigroup $(S(t))_{t\geq 0}$, In this case, $(U(t))_{t\in\mathbb{R}}$, is defined as follow:

$$U(t) = \begin{cases} T(t), & t \geq 0, \\ S(-t), & t \leq 0. \end{cases}$$

Theorem 1.3.1. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two $\mathcal{C}_0$-semigroups having as a generator the same operator A. Then*

$$T(t) = S(t),\ \forall t \geq 0.$$

Using the definition of a $\mathcal{C}_0$-semigroup and that of its generator, we can prove the following result:

Theorem 1.3.2. *Let $(T(t))_{t\geq 0}$ be a $\mathcal{C}_0$-semigroup on X of generator $(A, D(A))$, such that*

$$\|T(t)\| \leq Me^{\omega t}.$$

Then for $\lambda \in \mathbb{R}$, the operator $(A - \lambda I, D(A))$ is a generator of the $\mathcal{C}_0$-semigroup $(e^{-\lambda t}T(t))_{t\geq 0}$ on X.

1.3.3 Hille-Yosida theorem

In this subsection, we will present the theorem of Hille-Yosida which is a very powerful tool to decide whether a given operator is or is not the generator of some $\mathcal{C}_0$-semigroup. To prove this theorem, we will need some lemmas. Let us first introduce the Yosida approximation.

Definition 1.3.5. For $\lambda > 0$, we define the Yosida approximation of a linear operator A as follows:

$$A_\lambda = \lambda A R(\lambda, A),$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$. We have A_λ is a bounded operator. Indeed,

$$\begin{aligned} A_\lambda &= \lambda A R(\lambda, A) \\ &= \lambda[\lambda I - (\lambda I - A)]R(\lambda, A) \\ &= \lambda[\lambda R(\lambda, A) - I] \\ &= \lambda^2 R(\lambda, A) - \lambda I. \end{aligned}$$

Lemma 1.3.1. *Let A be a linear operator satisfying the conditions of Hille-Yosida theorem (see below). If A_λ is the Yosida approximation of A, then*

$$\lim_{\lambda \to +\infty} A_\lambda x = Ax, \quad \forall x \in D(A).$$

Lemma 1.3.2. *Let A be a linear operator satisfying the conditions of Hille-Yosida theorem (see below). If A_λ is the Yosida approximation of A, then A_λ is the generator of $\mathcal{C}_0$-semigroup of contraction $(e^{tA_\lambda})_{t\geq 0}$. Moreover, for all $x \in X$ and $\lambda, \mu > 0$, we have*

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq t\|A_\lambda x - A_\mu x\|.$$

Theorem 1.3.3. ***(Hille-Yosida theorem)***
Let $(T(t))_{t\geq 0}$ be a $\mathcal{C}_0$-semigroup on X. We say that the operator A (unbounded) is a generator of $(T(t))_{t\geq 0}$ if and only if

(1) A is closed

(2) $D(A)$ *is dense in* X

(3) $\rho(A) \supset [0, +\infty[$ *and there is* $M,\ \omega \in \mathbb{R}_+$ *such that*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(Re\lambda - \omega)^n},$$

for $n = 1, 2, ...,$ *if* $Re\lambda > \omega$.

Proposition 1.3.3. *Let* $(T(t))_{t\geq 0}$ *be a* $\mathcal{C}_0$*-semigroup such that*

$$\forall t \geq 0, ||T(t)||_{L(X)} \leq Me^{\omega t}\ (M \geq 1, \omega \geq 0).$$

Then A*, the generator of* $(T(t))_{t\geq 0}$*, verifies*

(1) $\rho(A) \supset \{\lambda \in \mathbb{C}, |Re\lambda| > \omega\}$ *and*

$$\forall \lambda \in \mathbb{C}, Re\lambda > \omega,\ \forall n \in \mathbb{N}^* : \|(\lambda I - A)^{-n}\|_{L(X)} \leq \frac{M}{(Re\lambda - \omega)^n}.$$

(2) The resolvent of A *is given by*

$$\forall \lambda \in \mathbb{C}, Re\lambda > \omega,\ (\lambda I - A)^{-1}x = \int_0^{+\infty} e^{-\lambda t}T(t)xdt.$$

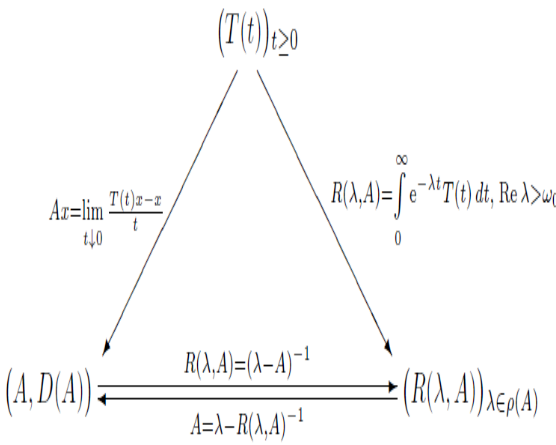
(3) The semigroup $(T(t))_{t\geq 0}$ *can be found from its generator* A *by*

$$T(t)x = \lim_{\lambda \longrightarrow +\infty} e^{tA_\lambda}x,\ t \geq 0,\ x \in X,$$

where $A_\lambda \in L(X)$ *is the Yosida approximation of* A *defined by*

$$A_\lambda = \lambda A(\lambda I - A)^{-1},\ \lambda > \omega.$$

The following diagram summarizes the relation between a $\mathcal{C}_0$-semigroup, its generator and its resolvent.



1.3.4 Analytic semigroup

In what follows, "arg" denotes the principal determination of the function argument characterized by

$$\arg(z)=\varphi \text{ if } z=re^{i\varphi},\ r>0,\ \varphi\in]-\pi,\pi].$$

Definition 1.3.6. For $\theta\in]0,\pi/2]$, consider the sector

$$\Sigma_\theta := \{z\in\mathbb{C}\setminus\{0\}\ : |arg(z)|<\theta\}.$$

Suppose that $T:\Sigma_\theta\cup\{0\}\longrightarrow\mathcal{L}(X)$ is a function with the following properties:

(i) $T:\Sigma_\theta\longrightarrow\mathcal{L}(X)$ is holomorphic.

(ii) For all $z,w\in\Sigma_\theta$, we have

$$T(z)T(w)=T(z+w),\ \text{and}\ T(0)=I.$$

(iii) For every $\theta'\in]0,\theta[$, the equality

$$\lim_{\substack{z\to 0\\ z\in\Sigma_{\theta'}}} T(z)f=f \text{ holds for all } f\in X,$$

then T is called an analytic semigroup of angle θ. Moreover, if

(iv) for all $\theta'\in]0,\theta[$, we have

$$\sup_{z\in\Sigma_{\theta'}}\|T(z)\|<+\infty,$$

then T is called a bounded analytic semigroup of angle θ. The generator of the restriction $T : [0, +\infty[\to \mathcal{L}(X)$ is called the generator of the analytic semigroup T.

Remark 1.3.2. Clearly, for an analytic semigroup T, the mapping

$$T :]0, +\infty[\to \mathcal{L}(X),\ t \longmapsto T(t) \in \mathcal{L}(X)$$

is continuous in the operator norm, it is even differentiable. Among others, this continuity has the following consequence: For λ sufficiently large, the resolvent of the generator is given by the improper integral

$$R(\lambda, A) = \int_0^{+\infty} e^{-\lambda t} T(t) dt,$$

which is convergent in the operator norm.

Proposition 1.3.4. *Let T be an analytic semigroup of angle $\theta \in]0, \frac{\pi}{2}]$ with generator A, then the following assertions are true:*

a) For every $r > 0$ and $\theta' \in]0, \theta[$, we have

$$\sup\{\|T(z)\| : z \in \Sigma_{\theta'},\ |z| \leq r\} < +\infty.$$

b) For all $\theta' \in]0, \theta[$, there exist $\omega = \omega_{\theta'} > 0$ and $M = M_{\theta'} \geq 1$ such that

$$\|T(z)\| \leq M e^{\omega Re(z)} \text{ for all } z \in \Sigma_{\theta'}.$$

c) For $\alpha \in] - \theta, \theta[$ and $t \geq 0$, define $T_\alpha(t) := T(e^{i\alpha} t)$, then T_α is a strongly continuous semigroup with generator $e^{i\alpha} A$.

Example 1.3.3. *For $A \in \mathcal{L}(X)$ and $z \in \mathbb{C}$ define*

$$T(z) = e^{zA} := \sum_{n=0}^{\infty} \frac{z^n A^n}{n!},$$

then T is an analytic semigroup.

Example 1.3.4. *The shift semigroup on $L^p(\mathbb{R})$ is not analytic. Or, more generally, if T is a strongly continuous group which is not continuous for the operator norm at $t = 0$, then T is not analytic.*

Theorem 1.3.4. *Let A be a closed linear operator with dense domain $D(A)$ in X and $0 < \theta \leq \frac{\pi}{2}$ such that*

$$\Sigma_{\frac{\pi}{2}+\theta} \subset \rho(A), \text{ and } \exists M > 0, \ \forall \lambda \in \rho(A), \ ||(A-\lambda)^{-1}|| \leq \frac{M}{|\lambda|}. \qquad (*)$$

We define $(T(t))_{t\geq 0}$, denoted by $(e^{tA})_{t\geq 0}$, by

$$T(0) = I \text{ and } \forall t > 0, \forall x \in X, \ T(t)x = e^{tA}x = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}(A - \lambda I)^{-1}x \, d\lambda,$$

where $\gamma \subset \rho(A)$ is an unbounded contour in $\Sigma_{\frac{\pi}{2}+\theta}$ from $+\infty e^{-i(\frac{\pi}{2}+\theta)}$ to $+\infty e^{i(\frac{\pi}{2}+\theta)}$.

Then $(e^{tA})_{t\geq 0}$ is an $\mathcal{C}_0$-semigroup of generator A. Moreover $(e^{tA})_{t\geq 0}$ is extended into an analytic semigroup of angle θ denoted by $(e^{zA})_{z\in\Sigma_\theta}$.

Remark 1.3.3. If A is a closed linear operator with dense domain $D(A)$ in X verifying (*) of the previous theorem, then $-A \in sect(\theta + \frac{\pi}{2})$.

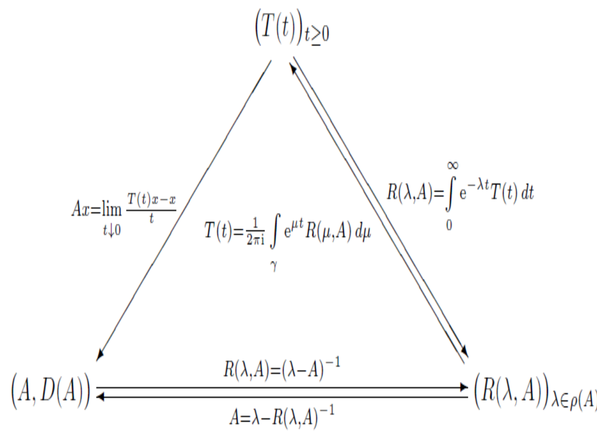
Theorem 1.3.5. *Let A be a closed linear operator with dense domain $D(A)$ in X such that*

$$]0, +\infty[\subset \rho(A), \text{ and } \exists M > 0, \ \forall \lambda > 0, \ ||(A-\lambda)^{-1}|| \leq \frac{M}{\lambda},$$

then there exists a sector Σ_ϕ, $0 < \phi \leq \frac{\pi}{2}$, such that

$$\Sigma_\phi \subset \rho(A), \text{ and } \exists M > 0, \ \forall \lambda \in \Sigma_\phi, \ ||(A-\lambda)^{-1}|| \leq \frac{M}{|\lambda|}.$$

The following diagram summarizes the relation between an analytic semigroup, its generator and its resolvent.



1.4 Interpolation

Let E_0 and E_1 be two Banach spaces and X be a separate topological space with

$$E_i \hookrightarrow X,\ i=0,1.$$

Consider the Banach spaces $E_0 \cap E_1$ and $E_0 + E_1$ equipped with the norms

$$\|a\|_{E_0\cap E_1} = \|a\|_{E_0} + \|a\|_{E_1},$$

and

$$\|a\|_{E_0+E_1} = \inf_{\substack{a=a_0+a_1\\ a_i\in E_i,\ i=0,1}} (\|a\|_{E_0} + \|a\|_{E_1}).$$

The couple (E_0, E_1) is called an interpolation couple.

Definition 1.4.1. Let (E_0, E_1) be an interpolation couple. We call intermediate space between E_0 and E_1, any Banach space E such that

$$E_0 \cap E_1 \hookrightarrow E \hookrightarrow E_0 + E_1.$$

Example 1.4.1. *The spaces $E_i,\ i=0,1$ are intermediate spaces.*

Theorem 1.4.1. ***(Marcel Riesz's theorem)***
Let $p_i, q_i \in [0,+\infty[$ and $\Omega_i,\ i=0,1$ be open sets of $\mathbb{R}^n$, and

$$K : L^{p_0}(\Omega_0) + L^{p_1}(\Omega_0) \to L^{q_0}(\Omega_1) + L^{q_1}(\Omega_1)$$

be a linear operator, such that

$$\begin{cases} K/_{L^{p_0}(\Omega_0)} \in \mathcal{L}\left(L^{p_0}, L^{q_0}\right), \\ K/_{L^{p_1}(\Omega_0)} \in \mathcal{L}\left(L^{p_1}, L^{q_1}\right). \end{cases}$$

Let $\theta \in [0,1]$ *and* $p_\theta, q_\theta \in [0, +\infty[$, *such that*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then

$$K/_{L^{p_\theta}(\Omega_0)} \in \mathcal{L}\left(L^{p_\theta}, L^{q_\theta}\right).$$

Moreover

$$||K||_{L(L^{p_\theta}, L^{q_\theta})} \leq ||K||^{1-\theta}_{L(L^{p_0}, L^{q_0})} \cdot ||K||^{\theta}_{L(L^{p_1}, L^{q_1})}.$$

Lemma 1.4.1. *(Schur's lemma)*

Let $k : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$ *be a measurable function such that*

i) $\exists a > 0, \forall x_2 \in \Omega_2 : \int_{\Omega_1} |k(x_1, x_2)|\, dx_1 \leq a$,

ii) $\exists b > 0, \forall x_1 \in \Omega_1 : \int_{\Omega_2} |k(x_1, x_2)|\, dx_2 \leq b$.

We define the operator K *by*

$$(Kf)(x_2) = \int_{\Omega_1} k(x_1, x_2) f(x_1)\, dx_1; \forall x_2 \in \Omega_2.$$

Then, $\forall p \in [1, +\infty]$:

$$K \in \mathcal{L}\left(L^p(\Omega_1), L^p(\Omega_2)\right).$$

Proof. We have $K \in \mathcal{L}\left(L^1(\Omega_1), L^1(\Omega_2)\right)$ and $K \in \mathcal{L}\left(L^\infty(\Omega_1), L^\infty(\Omega_2)\right)$, then thanks to the theorem of Marcel Riesz we obtain the result. □

Interpolation Spaces

Definition 1.4.2. An intermediate space E between E_0 and E_1 is an interpolation space between E_0 and E_1, if for each $K \in L(E_0 + E_1)$:

$$K \in \mathcal{L}(E_i),\ i = 0, 1 \Longrightarrow K \in \mathcal{L}(E).$$

Definition 1.4.3. Let (E_0, E_1) and (F_0, F_1) be two interpolation couples. We say that two Banach spaces E and F are interpolation spaces between (E_0, E_1) and (F_0, F_1), if

1. E is intermediate space between E_0 and E_1,
2. F is intermediate space between F_0 and F_1,
3. $K \in \mathcal{L}(E_i, F_i), i = 0, 1 \Longrightarrow K \in \mathcal{L}(E, F)$.

Example 1.4.2.

1. *$E_0 \cap E_1$ is an interpolation space between E_0 and E_1.*
2. *$E_0 + E_1$ is an interpolation space between E_0 and E_1.*

Theorem 1.4.2. ***(Fundamental property of interpolation)***
Let E and F be two interpolation spaces between (E_0, E_1) and (F_0, F_1), then there exists $C \geq 0$ such that for each K:

$$K \in L(E_0 + E_1, F_0 + F_1) \text{ and } K \in \mathcal{L}(E_i, F_i), i = 0, 1,$$

we have

$$\|K\|_{\mathcal{L}(E,F)} \leq C \max_{i=0,1}(\|K\|_{\mathcal{L}(E_i,F_i)}).$$

Definition 1.4.4. Two interpolation spaces E and F are said to have exponent $\theta \in [0, 1]$, if there exists a constant $C \geq 0$ such that for each K:

$$\|K\|_{\mathcal{L}(E,F)} \leq C\|K\|^{1-\theta}_{\mathcal{L}(E_0,F_0)} \cdot \|K\|^{\theta}_{\mathcal{L}(E_1,F_1)}.$$

Theorem 1.4.3. *Let (E_0, E_1) be an interpolation couple. Let $\theta \in]0, 1[$ and $p \in [1, \infty]$, then the space $(E_0, E_1)_{\theta,p}$ defined by*

$$x \in (E_0, E_1)_{\theta,p} \Longleftrightarrow \begin{cases} i)\ \forall t > 0,\ \exists u_0(t) \in E_0,\ \exists u_1(t) \in E_1 : x = u_0(t) + u_1(t), \\ ii)\ t^{-\theta}u_0 \in L^p_*(\mathbb{R}_+, E_0),\ t^{1-\theta}u_1 \in L^p_*(\mathbb{R}_+, E_1), \end{cases}$$

(where L^p_* denotes the space of p-integrable functions with the measure $\frac{dt}{t}$),
equipped with the norm

$$\|x\|_{\theta,p} = \inf_{u_0,u_1} \left(\|t^{-\theta} u_0\|_{L^p_*(\mathbb{R}_+,E_0)} + \|t^{1-\theta} u_1\|_{L^p_*(\mathbb{R}_+,E_1)} \right),$$

is an interpolation space between E_0 and E_1.

Definition 1.4.5. Let A be a closed linear operator and its domain $D_A \subset E$ is equipped with the graph norm

$$\forall x \in D_A, \|x\|_{D_A} = \|x\|_X + \|Ax\|_X.$$

Then we set, following the notations of P.Grisvard,

$$D_A(\theta, p) = (D_A, E)_{1-\theta,p} \text{ where } p \in [1, +\infty] \text{ and } 0 < \theta < 1.$$

When operator A verifies some additional assumptions, it is then possible to give explicit characterizations of $D_A(\theta, p)$, thus:

Theorem 1.4.4. *Let $p \in [1, +\infty]$ and $\theta \in]0, 1[$.*

1. *Suppose that $\rho(A) \supset]0, +\infty[$ and there exists a constant $C > 0$ such that*

$$\forall \lambda > 0, \; \left\|(A - \lambda I)^{-1}\right\|_{L(E)} \leq \frac{C}{\lambda}.$$

Then,

$$D_A(\theta, p) = \left\{x \in E : t^{\theta} A(A - tI)^{-1} x \in L^p_* (\mathbb{R}_+, E)\right\}.$$

2. *If A generates a strongly continuous bounded semigroup in E, then*

$$D_A(\theta, p) = \left\{x \in E : t^{-\theta} \left(e^{At} - I\right) x \in L^p_* (\mathbb{R}_+, E)\right\}.$$

3. *If A generates a bounded analytic semigroup in E, then*

$$D_A(\theta, p) = \left\{x \in E : t^{1-\theta} A e^{At} x \in L^p_* (\mathbb{R}_+, E)\right\}.$$

So, if operators A and B are sectorial, then the intermediate spaces $D_A(\theta,p)$ between D_A and E (or D_B and E) are characterized by:

$$D_A(\theta,p)=\left\{x\in E:t^{\theta}A(A-tI)^{-1}x\in L_*^p\left(\mathbb{R}_+,E\right)\right\}.$$

In particular

$$D_A(\theta,+\infty)=\left\{x\in E:\sup_{r>0}r^{\theta}\left\|A(A-r)^{-1}x\right\|<+\infty\right\}.$$

Hereafter, we specify some interpolation spaces.

Example 1.4.3.

1. *Let* $E=C\left([0,1],\|\cdot\|_{\infty}\right)$ *and the operator* A *defined by :*

$$\begin{cases} D_A=\left\{\varphi\in C^2([0,1]):\varphi(0)=\varphi(1)=0\right\}, \\ A\varphi=\varphi''. \end{cases}$$

So for $p=+\infty$

$$D_A(\theta,+\infty)=\begin{cases} C^{2\theta}([0,1]) \text{ and } \varphi(0)=\varphi(1)=0 \text{ if } 2\theta<1, \\ C^{1,*}([0,1]) \text{ and } \varphi(0)=\varphi(1)=0, \\ C^{1,2\theta-1}([0,1]) \text{ and } \varphi(0)=\varphi(1)=0, \end{cases}$$

where $C^{1,*}([0,1])$ *is the so-called Zygmund space of continuous functions* φ *on* $[0,1]$ *such that*

$$\sup_{x,y\in[0,1]}\frac{\left|\varphi(x)-2\varphi\left(\frac{x+y}{2}\right)+\varphi(y)\right|}{|x-y|}<+\infty.$$

2. *Let* $E=L^p(]0,1[),p\in[1,+\infty[$ *and the operator* B *defined by*

$$\begin{cases} D_B=\left\{u\in W^{1,p}(]0,1[)/u(0)=0\right\}, \\ Bu=u', \end{cases}$$

then

$$D_B(\theta,p)=\begin{cases} W^{\theta,p}(]0,1[) \text{ if } 0<\theta<\frac{1}{p}, \\ W_{0,0}^{\theta,p}(]0,1[) \text{ if } \theta=\frac{1}{p}, \\ W_0^{\theta,p}(]0,1[) \text{ if } \frac{1}{p}<\theta<1. \end{cases}$$

We recall that

- *For* $\theta \in]0.1\left[,\ W^{\theta,p}(]0.1[)\right.$ *is the subspace of* $L^p(]0.1[)$ *of functions* u *such that*

$$\int_0^1 \int_0^1 \frac{|u(t)-u(s)|^p}{|t-s|^{1+\theta p}} dtds < +\infty.$$

For $p = 2,\ \theta = \frac{1}{2}$, *we find the famous* $H^{\frac{1}{2}}(]0,1[)$ *space.*

- *The Sobolev space (rather called Besov space)* $W_{0,0}^{\theta,p}(]0,1[)$ *is the subspace of the functions* u *of* $L^p(]0.1[)$ *such that*

$$\int_0^1 \frac{|u(t)|^p}{t} dt < +\infty.$$

- *In general, we can give*

$$(W^{m,p}(\Omega), L^p(\Omega))_{\theta,q} = \mathcal{B}_{p,q}^{m(1-\theta)}(\Omega),$$

where Ω *is an open set with boundary of class* C^m, $p \in]1,+\infty[$, $q \in [1,+\infty]$, *and* $\mathcal{B}_{p,q}^s(\Omega)$ *are the Besov spaces.*

Properties

We now give some fundamental properties of interpolation spaces. For all $\theta \in]0,1[$ and $p, q \in [1,+\infty]$, we have

1. If $0 < \omega \leq \theta < 1$ and $p, q \in [1,+\infty]$, then

$$(E_0, E_1)_{\theta,p} \hookrightarrow (E_0, E_1)_{\omega,q}.$$

2. If $p \leq q$, then

$$(E_0, E_1)_{\theta,p} \hookrightarrow (E_0, E_1)_{\theta,q}.$$

3. If $E_0 = E_1$, then $(E_0, E_1)_{\theta,p} = E_0 = E_1$.

4. $(E_0, E_1)_{\theta,p} = (E_1, E_0)_{1-\theta,p}$.

Particular Case $(D_A, E)_{\theta,p}$

Let A be a closed linear operator in E. We set $E_0 = D_A$ which is equipped with the graph norm and $E_1 = E$. Then $E_0 \cap E_1 = D_A$ and $E_0 + E_1 = E$. So,

$$D_A \hookrightarrow (D_A, E)_{\theta,p} \hookrightarrow E.$$

Theorem 1.4.5. *If A is the infinitesimal generator of a strongly continuous semigroup $G(t)$, then*

$$x \in (D_A, E)_{\theta,p} \Longleftrightarrow \begin{cases} i) x \in E, \\ \quad ii) \ \frac{G(t)x - x}{t^{1-\theta}} \in L^p_*(E), \end{cases}$$

with $p \in [1, +\infty]$. Or, equivalently

$$(D_A, E)_{\theta,p} = \left\{ x \in E : \left\| t^{\theta-1}(G(t) - I)x \right\|_E \in L^p_* \right\},$$

which is equipped with the norm

$$\|x\|_{(D_A,E)_{\theta,p}} = \|x\|_E + \left(\int_0^{+\infty} t^{-(1-\theta)p} \|G(t - I)x\|^p \frac{dt}{t} \right)^{\frac{1}{p}},$$

and with the usual modifications if $p = \infty$.

Chapter 2

Sum of linear operators' method

The theory of linear operators' sum is concerned with the study of the spectral properties of the sum's operator $A+B$ from those of the linear operators A and B. It is marked by two important dates: In 1975, G. Da Prato and P. Grisvard (see [7]) unify their earlier results to develop a remarkable theory that now bears their name. In 1987, G. Dore and A. Venni, in a famous paper (see [9]), give optimal results on the sum's theory in the framework of UMD spaces (Unconditional Martingale Differences). Here, we will only deal with the first approach applicable to the study of parabolic type equations. The commutative case will be exposed in the first section, and the non-commutative case that we will use in the third chapter, will be exposed in the second section. The sum of linear operators' method is based on an explicit construction of the solution in the form of a Dunford integral and on the use of interpolation spaces characterized by Grisvard [34].

2.1 Commutative case

2.1.1 Introduction

In the sequel, X denotes a complex Banach space. Let A and B be two closed linear operators in X with domains $D(A)$ and $D(B)$, respectively. We are then interested by the resolution of the following equation:

$$Au + Bu = f, \tag{2.1}$$

where f is a given vector of X and u is the unknown. The sum's operator $L = A + B$ is defined by

$$\begin{cases} D(L) = D(A) \cap D(B) \\ Lu = Au + Bu \text{ if } u \in D(L), \end{cases}$$

and the equation (2.1) becomes

$$Lu = f.$$

A strict solution of the equation (2.1) is an element $u \in D(L)$ satisfying the equation (2.1). The ideal is to find such a solution when f is an arbitrary element of X, but this is not always possible. We therefore introduce a new notion of solution. Namely, u is a strong solution of equation (2.1) if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}} \in D(L)$ such that

$$\lim_{n\to+\infty} u_n = u \text{ and } \lim_{n\to+\infty} Lu_n = f. \tag{2.2}$$

Obviously, a strict solution of the equation (2.1) is a strong solution of the equation (2.1). The notion of strong solution is therefore weaker (but the term weak solution will not be used here, it is generally reserved for variational solutions).

Note that if L is closed, the two notions of strict and strong solution are equivalent, but the sum of two closed operators is not necessarily closed. On the other hand if we assume that L is closable then (2.1) is equivalent to

$$u \in D(\bar{L}) \text{ and } \bar{L}u = f.$$

Finally in the case where L is closable, the following propositions are equivalent:

1. For every f of X, there exists a strong solution of (2.1).

2. $0 \in \rho(\bar{L})$.

And if L is closed, the following propositions are equivalent:

1. for all f of X, there is a strict solution of (2.1).

2. $0 \in \rho(L)$.

In this context, we understand the importance of finding reasonable conditions on the operators A and B, which ensure that L is closable (even closed) and that $0 \in \rho(\bar{L})$. Otherwise, we can introduce a spectral parameter λ and consider the equation

$$Au + Bu - \lambda u = f.$$

The theorems of G. Da Prato and P. Grisvard, stated later, give positive answers to these problems on the sums of operators.

2.1.2 Assumptions on A and B

Let A and B be two closed linear operators in X, with respective domains $D(A)$ and $D(B)$. We propose to solve, for positive λ, the equation

$$Au + Bu - \lambda u = f. \tag{2.3}$$

This is equation (2.1) where A is replaced by $A - \lambda I$. We recall that arg denotes the principal determination of the function argument characterized by

$$\arg(z) = \varphi \text{ if } z = re^{i\varphi}, r > 0, \varphi \in]-\pi, \pi].$$

We define for $\theta \in]0, \pi[$, the sector

$$\Sigma_{\pi-\theta} = \left\{z \in \mathbb{C}^* : |\arg(z)| < \pi - \theta\right\},$$

and a closed linear operator P on X is said to satisfy the hypothesis (H_θ) if and only if

$$\left\{\begin{array}{l} \rho(P) \supset \Sigma_{\pi-\theta} \text{ and there is a convex even function} \\ C_P :]-\pi+\theta, \pi-\theta[\longrightarrow \mathbb{R} \text{ such that:} \\ \|(A - zI)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_P(\phi)}{|z|} \text{ for all } z \in \Sigma_{\pi-\theta} \text{ such that } \arg z = \phi. \end{array}\right. \tag{H_θ}$$

The two basic hypotheses on the operators A and B are the following:

$$(Parabolicity) \left\{\begin{array}{l} \exists R > 0, \exists \theta_A, \theta_B \in]0, \pi[: \\ \text{1) } A \text{ verifies } (H_{\theta_A}), B \text{ verifies } (H_{\theta_B}) \\ \text{2) } \theta_A + \theta_B < \pi. \end{array}\right. \tag{DP1}$$

and

$$\left\{\begin{array}{l} \forall \lambda \in \rho(A), \forall \mu \in \rho(B) \\ (A - \lambda I)^{-1}(B - \mu I)^{-1} = (B - \mu I)^{-1}(A - \lambda I)^{-1}. \end{array}\right. \tag{DP2}$$

Remark 2.1.1. 1. Since $\theta_A + \theta_B < \pi$, one of the two angles is therefore strictly less than $\frac{\pi}{2}$ and the corresponding operator is then the infinitesimal generator of an analytic semigroup (not necessarily continuous at 0 , the domains $D(A)$ and $D(B)$ not having been assumed to be dense).

2. Under the assumptions (DP2), if $y \in X, \lambda \in \rho(A)$ and $\mu \in \rho(B)$, then

$$(A - \lambda I)^{-1}(B - \mu I)^{-1}y = (B - \mu I)^{-1}(A - \lambda I)^{-1}y \in D(A) \cap D(B).$$

Moreover if $x \in D(B)$, even considering $y = (B - \mu I)x$ we obtain

$$\begin{cases} (A - \lambda I)^{-1}x \in D(B) \text{ and} \\ (B - \mu I)(A - \lambda I)^{-1}x = (A - \lambda I)^{-1}(B - \mu I)x. \end{cases}$$

The hypothesis (DP2) will allow us to commute the products in any functional expression containing $A,\ B$ and I. In the sequel, the following commutations will be used:

Under the hypothesis (DP2), we consider $\lambda + z \in \rho(A)$ and $-z \in \rho(B)$.

1. If $f \in X$, then

$$(A - \lambda - z)^{-1}(B + z)^{-1}f \in D(B),$$

and

$$B(A - \lambda - z)^{-1}(B + z)^{-1}f = (A - \lambda - z)^{-1}B(B + z)^{-1}f.$$

2. If moreover $f \in D(B)$, then

$$B(A - \lambda - z)^{-1}(B + z)^{-1}f = (A - \lambda - z)^{-1}(B + z)^{-1}Bf.$$

The Sectorial Curve γ_λ

The hypotheses (DP1) and (DP2) will allow us to construct a sectorial curve γ_λ separating $\sigma(A - \lambda I)$ and $\sigma(-B)$ and remaining in $\rho(A - \lambda I) \cap \rho(-B)$. So there exists $\theta_0 > 0$ such that $\theta_B < \theta_0 < \pi - \theta_A$. Note that under the hypothesis (DP1) one of the two angles θ_A, θ_B is smaller than $\frac{\pi}{2}$. So, in the sequel, we will assume that

$$\theta_A < \frac{\pi}{2} \text{ and } \theta_0 > \frac{\pi}{2}.$$

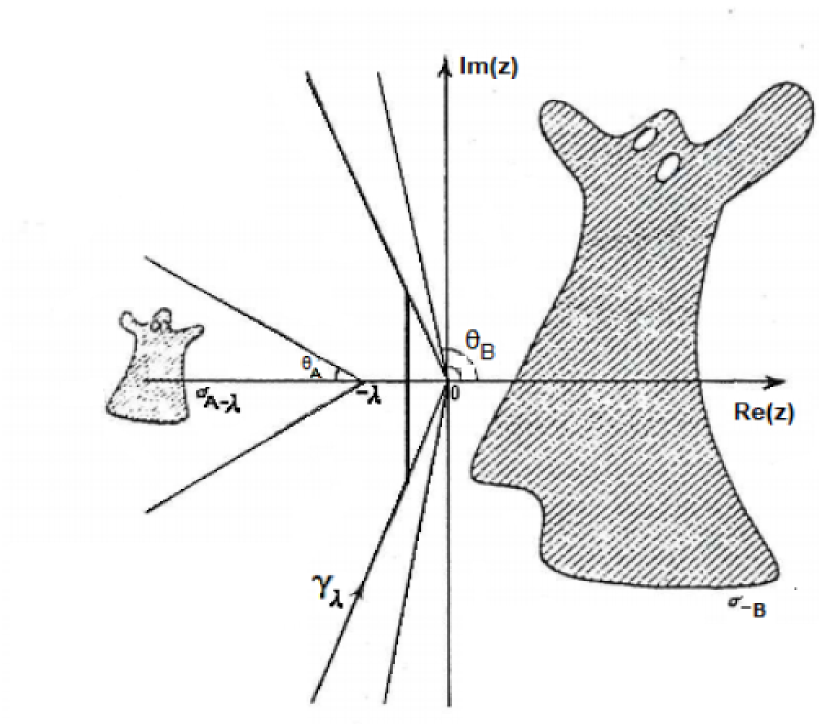
We now construct, for all $\lambda > 0$, an infinite sectorial curve γ_λ which is the positively oriented boundary of the domain located at the left of the lines

$$\left\{te^{i\theta_0} : t \geqslant 0\right\}, \left\{te^{-i\theta_0} : t \geqslant 0\right\}, \left\{-\frac{\lambda}{2} + y; y \in \mathbb{R}\right\},$$

and who lives in $\rho(A-\lambda I) \cap \rho(-B)$. This curve is oriented, leaving the $\sigma(A-\lambda I)$ spectrum on the left (see figure below).

We can then notice that $z \mapsto (A-\lambda-z)^{-1}$ is defined and analytic to the right of γ_λ and that $z \mapsto (B+z)^{-1}$ is defined and analytic to the left of γ_λ. Note that

$$\rho(A-\lambda I) \cup \rho(-B) = \mathbb{C}.$$



2.1.3 Representation of the solution

Our aim is to solve the equation

$$Au + Bu - \lambda u = f, \; \lambda > 0,$$

under the hypotheses (DP1) and (DP2). This is to see when it is possible to define $(A+B-\lambda I)^{-1}$ or, failing that, $(\overline{A+B} - \lambda I)^{-1}$.

Note that the hypothesis of ellipticity-parabolicity (DP1) alone is not sufficient to guarantee that the sum $A+B$ is closable. Similarly, if the two hypotheses (DP1) and (DP2) are verified, this does not ensure that $A+B$ is closed. On the other hand, they are sufficient to have the closability of the sum even if neither of the two domains $D(A), D(B)$ is dense in X, see [21].There are other additional hypotheses, among others, allowing to obtain the closedness of the sum $A+B$, see [9].

So what is the "candidate" to be $(\overline{A+B}-\lambda I)^{-1}$ or $(A+B-\lambda)^{-1}$? We will build it using the Dunford integral (cf. chapter 1):

$$g(T)=\frac{1}{2i\pi}\int_{\Gamma} g(z)(zI-T)^{-1}dz.$$

Here, let's put

$$\begin{cases} g(z)=(B+zI)^{-1} \\ T=A-\lambda I=\mathrm{c}^{\mathrm{ste}} \text{ operator,}\end{cases}$$

then

$$\begin{aligned} g(A-\lambda I) &= \frac{1}{2i\pi}\int_{\Gamma} g(z)(zI-(A-\lambda I))^{-1}dz \\ &= -\frac{1}{2i\pi}\int_{\Gamma}(A-\lambda I-zI)^{-1}(B+zI)^{-1}dz, \end{aligned}$$

where Γ would be a Jordan curve surrounding the spectrum of $A-\lambda I$. This leads to consider, in what follows, the operator

$$f\in X\longmapsto S_\lambda f=-\frac{1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda I-zI)^{-1}(B+zI)^{-1}f dz.$$

Note that for $z\in\gamma_\lambda$, the two resolvents written above exist thanks to (DP1). The following proposition shows that S_λ is a bounded linear operator on X.

Proposition 2.1.1. *Let A and B be two closed linear operators in X satisfying (DP1) and (DP2). So, for all $\lambda>0, S_\lambda\in\mathcal{L}(X)$ and there exists $C>0$:*

$$\|S_\lambda\|_{\mathcal{L}(X)}\leqslant\frac{C}{\lambda}.$$

The following proposition represents a fundamental results in the theory of sums.

Proposition 2.1.2. *Let $\lambda>0$*

1. *If $u \in D(A+B) = D(A) \cap D(B)$, then*

$$S_\lambda(Au + Bu - \lambda u) = u,$$

(So S_λ is a left inverse of $(A + B - \lambda I)$ on $D(A+B)$).

2. *If $f \in D(A) + D(B)$, then*

$$S_\lambda f \in D(A) \cap D(B) \text{ and } (A + B - \lambda I) S_\lambda f = f,$$

(S_λ is therefore also a right inverse of $(A + B - \lambda I)$ but on $D(A)$ or $D(B)$).

We have just seen that if $f \in D(A) + D(B)$ then $u = S_\lambda f$ is a solution of (2.3). Can we do better, i.e. to obtain an inverse of the sum assuming less regularity on f? The ideal is to solve equation (2.3) for $f \in X$. This requires the use of interpolation spaces (*cf.* chapter 1, [14]).

2.1.4 Main theorems

We come back to our equation (2.3): $Au + Bu - \lambda u = f,\ \lambda > 0$.

Theorem 2.1.1. *(Strict solution)*
Let A and B be two closed linear operators in X verifying (DP1) and (DP2). Then, for all $f \in D_B(\theta, +\infty) + D_A(\theta, +\infty)$ where $\theta \in]0,1[$, there exists a unique strict solution u of

$$Au + Bu - \lambda u = f.$$

Moreover u is given by the Dunford integral

$$u = S_\lambda f = -\frac{1}{2i\pi} \int_{\gamma_\lambda} (A - \lambda - zI)^{-1} (B + zI)^{-1} f dz.$$

Proof. We assume in the following that $f \in D_B(\theta, +\infty)$, for example. Let us show that $u + S_\lambda f \in D(B)$. We have

$$(A - \lambda - z)^{-1}(B + z)^{-1} f = (B + z)^{-1}(A - \lambda - z)^{-1} f,$$

$$B(A - \lambda - z)^{-1}(B + z)^{-1} f = (A - \lambda - z)^{-1} B(B + z)^{-1} f,$$

and

$$\left\|B(A-\lambda-z)^{-1}(B+z)^{-1}f\right\|_X \leqslant \frac{\text{Cste}}{|z+\lambda||z|^\theta}\|f\|_{D_B(\theta,+\infty)}.$$

Thus

$$-\frac{1}{2i\pi}\int_{\gamma_\lambda} B(A-\lambda-z)^{-1}(B+z)^{-1}f dz,$$

is absolutely convergent and $u \in D(B)$ with

$$Bu = -\frac{1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}B(B+z)^{-1}f dz.$$

Similarly, we can show that $u \in D(A)$ as well as

$$Au + Bu - \lambda u = f.$$

$\square$

We have just seen that to obtain a strict solution u of (2.3), we have given ourselves a regularity on f, namely

$$f \in D_B(\theta,+\infty) + D_A(\theta,+\infty).$$

So obviously the block $Au + Bu - \lambda u$ is also in $D_B(\theta,+\infty)+$ $D_A(\theta,+\infty)$. The natural question then arises to know if, in the end, we recover on each of the terms of the block, the regularity that we gave ourselves at the start on f. The answer to this question is positive as shown by the following result.

Theorem 2.1.2. *(Maximal regularity)*
Let A and B be two closed linear operators in X satisfying $(DP1)$ and (DP2). For all $f \in D_A(\theta,+\infty)$, the strict solution u of

$$Au + Bu - \lambda u = f,$$

verifies

1. $(A-\lambda)u \in D_A(\theta,+\infty)$,

2. $Bu \in D_A(\theta,+\infty)$,

3. $(A-\lambda)u \in D_B(\theta,+\infty)$.

We have, by analogy, the following theorem.

Theorem 2.1.3. *(Maximal regularity)*
Let A and B be two closed linear operators in X satisfying $(DP1)$ and (DP2). For all $f \in D_B(\theta,+\infty)$, the strict solution u of

$$Au + Bu - \lambda u = f,$$

verifies

1. $(A-\lambda)u \in D_B(\theta,+\infty)$,
2. $Bu \in D_B(\theta,+\infty)$,
3. $Bu \in D_A(\theta,+\infty)$.

Proof. (of the first theorem) Let $t>0$ be large enough so that $-t$ is to the left of point $\frac{-\lambda}{2}$. So

$$(-B-(-t))^{-1} \in \mathcal{L}(X),$$

i.e.

$$(B-t)^{-1} \in \mathcal{L}(X).$$

Let us then show the regularity $Bu \in D_B(\theta,+\infty)$. We have to calculate $(B-t)^{-1}u$. We have

$$\begin{aligned}(B-t)^{-1}u &= (B-t)^{-1}(S_\lambda f)\\ &= \frac{-1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}(B-t)^{-1}(B+z)^{-1}f dz.\end{aligned}$$

We use the 2$^{\text{nd}}$ identity of the resolvent:

$$(B-t)^{-1}(B+z)^{-1}f = \frac{1}{t+z}\left((B-t)^{-1}-(B+z)^{-1}\right),$$

(similar to a decomposition into simple elements). It comes:

$$(B-t)^{-1}u = \frac{-1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}(B-t)^{-1}f\frac{dz}{t+z} + \frac{1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}(B+z)^{-1}f\frac{dz}{t+z}.$$

The first integral is zero (integrate to the right of $\gamma_\lambda, t+z \neq 0$). From where

$$\begin{aligned}
&B(B-t)^{-1}u \\
=&u+t(B-t)^{-1}u \\
=&\frac{-1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}(B+z)^{-1}f dz \\
&+\frac{1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}(B+z)^{-1}\frac{(tf)}{t+z}dz \\
=&\frac{-1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}\left\{(B+z)^{-1}f-\frac{t}{t+z}(B+z)^{-1}f\right\}dz \\
=&\frac{-1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}(B+z)^{-1}\left(\frac{t}{t+z}\right)f dz.
\end{aligned}$$

By doing the same work for $Bu=B\left(S_\lambda f\right)$, we have for t large enough

$$\begin{aligned}
B(B-t)^{-1}Bu &= \frac{-1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}B(B+z)^{-1}\frac{zf}{t+z}dz \\
&= \frac{-1}{2i\pi}\int_{\gamma_\lambda}\frac{z}{t+z}(A-\lambda-z)^{-1}B(B+z)^{-1}f dz.
\end{aligned}$$

So

$$\begin{aligned}
\left\|B(B-t)^{-1}Bu\right\|_X &= (\text{ Cste })\int_{\gamma_\lambda}\left|\frac{z}{\lambda+z}\right|\frac{1}{|t+z|}\frac{1}{|z|^\theta}|dz|\|f\|_{D_B(\theta,\infty)} \\
&\leqslant (\text{ Cste })\int_{\gamma_\lambda}\frac{1}{|t+z|}\frac{1}{|z|^\theta}|dz|\|f\|_{D_B(\theta,\infty)},
\end{aligned}$$

then $z\to tz$

$$\left\|B(B-t)^{-1}Bu\right\|_X \leqslant (\text{ Cste })\left[\int_{\gamma_\lambda}\frac{1}{|1+z|}\frac{1}{|z|^\theta}|dz|\right]\frac{\|f\|_{D_B(\theta,\infty)}}{t^\theta}.$$

From where

$$t^\theta\left\|B(B-t)^{-1}Bu\right\|_X \leqslant (\text{ Cste })\|f\|_{D_B(\theta,\infty)},$$

i.e. $Bu\in D_B(\theta,+\infty)$. So since

$$u\in D(B)\subset D_B(\theta,+\infty),$$

we deduce from the equation $Au+Bu-\lambda u=f$, that

$$Au\in D_B(\theta,+\infty).$$

$\square$

Finally the results obtained here extend to the case $f \in D_B(\theta, p)$ with $p \in [1, +\infty|$ thanks to the fundamental property of interpolation and to the following inclusions:

$$\begin{cases} D_A\left(\theta', q\right) \hookrightarrow D_A(\theta, p) \\ \text{if } \theta' > \theta,\ p,\ q \text{ any} \\ \text{or if } \theta' = \theta,\ q \leq p. \end{cases}$$

Proposition 2.1.3. *Let A and B be two closed linear operators in X verifying (DP1) and (DP2). If $f \in D_B(\theta, p)$, $\theta \in]0, 1[$, $p \in [1, +\infty[$ then the equation $Au + Bu - \lambda u = f$, admits a unique strict solution $u\,(u = S_\lambda f)$ satisfying*

1. $(A - \lambda)u \in D_B\left(\theta, +\infty\right)$,

2. $Bu \in D_B(\theta, +\infty)$,

3. $Bu \in D_A(\theta, +\infty)$.

2.2 Non-commutative case

2.2.1 Introduction

The method, as we have said earlier, is essentially based on an explicit construction of the solution in the form of a Dunford integral and on the use of interpolation spaces characterized by Grisvard [15] (see chapter 1). More precisely, let X be a complex Banach space and A and B be two closed linear operators of domains D_A and D_B respectively, not necessarily dense; we then consider the equation

$$Aw + Bw - \lambda w = y, \quad \lambda > 0. \tag{2.4}$$

We will say that the equation (2.4) is of parabolic type if

$$\begin{cases} (A - z)^{-1} \text{ and } (B - z)^{-1} \text{ exist for } z \in \Sigma_A \text{ and } z \in \Sigma_B, \text{where} \\ \sum_A \text{ and } \Sigma_B \text{ are two sectors of the form } \{z \in \mathbb{C}/|\arg z| < \\ < \pi - \varphi\} \text{ with } \varphi = \theta_A \text{ and } \varphi = \theta_B \text{ respectively and} \\ \theta_A + \theta_B < \pi \text{ and} \\ \|(A - z)^{-1}\|_{\mathcal{L}(X)} = 0\left(\frac{1}{|z|}\right) \text{ and } \|(B - z)^{-1}\|_{\mathcal{L}(X)} = 0\left(\frac{1}{|z|}\right) \\ \text{ for } z \in \Sigma_A \text{ and } \Sigma_B \text{ respectively.} \end{cases} \tag{H.1}$$

The commutative case corresponding to $[(A-\lambda)^{-1};(B-\mu)^{-1}]=0$ was treated in the previous section. Indeed the solution x has been constructed in the form

$$x=S_\lambda y=-\frac{1}{2i\pi}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}(B+z)^{-1}ydz, \tag{2.5}$$

where γ_λ is a simple curve joining $\infty\exp[i\theta_0]$ to $\infty\exp[-i\theta_0]$ (where $\theta_0\in]\theta_B,\pi-\theta_A[$) and residing in $\Sigma_{A-\lambda}\cap\Sigma_{-B}$. (defined as in the previous section). The fundamental idea here in this section is to construct the solution in the form of a "left inverse" for $(A+B-\lambda)$, more precisely we look for x in the form $x=(1+T_\lambda)^{-1}S_\lambda y$ where T_λ is "small enough" and zero in the commutative case. We make the assumption:

$$\left\{\begin{array}{l}\exists\lambda_0>0,\quad \psi:]0,+\infty[\times]0,+\infty[\to\mathbb{R}\text{ such that}\\ \left\|z'\left(A-\lambda_0\right)\left(A-z\right)^{-1}\left[\left(A-\lambda_0\right)^{-1};\left(B+z'\right)^{-1}\right]\right\|_{\mathcal{L}(X)}\leqslant\psi\left(|z|,|z'|\right)\\ \forall z\in\Sigma_A,\forall z'\in\Sigma_{-B}\text{ and }\int_{\gamma_\lambda}\psi(|z+\lambda|,|z|)|dz|\underset{\lambda\to+\infty}{\longrightarrow}0\,.\end{array}\right. \tag{H.2}$$

The hypotheses (H.1) and (H.2) and λ large enough make it possible to construct a unique solution for the problem (2.4) for a second member y in an interpolation space between D_A and X or D_B and X. To study the maximal regularity of the solution we need to make explicit the function ψ of (H.2); we will then show that if the data y belongs to an interpolation space $D_A(\theta,p)$ (see [34] or chapter 1) then Ax and Bx are in $D_A(\min(\delta,\theta),p)$ where δ will be given explained in the next paragraph. The part of the theory of sums dealt with in this section is applicable to the study of equations of parabolic type. The main theorems of this part can be found in [20] and [21].

2.2.2 Hypotheses

Let X be a complex Banach space and A and B be two closed linear operators with domains D_A and D_B, respectively, not necessarily dense in X. The operator $A+B$ will be denoted by L and it will be defined by $Lx=Ax+Bx$ for $x\in D_L=D_A\cap D_B$. We propose to solve the abstract equation:

$$\left\{\begin{array}{l}Lx-\lambda x=y,\quad \lambda>0,\\ x\in D_A\cap D_B.\end{array}\right. \tag{2.6}$$

We will say that a linear map P of domain $D_P\subset X$ verifies $H(\varphi)$ if there exists $\varphi\in[0,\pi[$ such that:

1. $\rho_P \subset \Sigma_P = \{z \in C / |\arg z| < \pi - \varphi\}$,

2. there exists an even and convex numerical function C_P defined on $]-\pi+\varphi;\pi-\varphi[$ such that:
$$\left\|(P-z)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{C_P(\theta)}{|z|}, \quad \forall z \in \rho_P,$$
with $\arg z = \theta$.

We will make the following assumptions:

Hypothesis (H.1): $\exists \theta_A \geqslant 0, \exists \theta_B \geqslant 0$ such that A verifies $H(\theta_A)$, B verifies $H(\theta_B)$, and $\theta_A + \theta_B < \pi$.

Hypothesis (H.2): Operators A and B verify (H.1) and in addition:
$$\exists \lambda_0 > 0, \quad \psi :]0,+\infty[\times]0,+\infty[\to \mathbb{R}_+$$
such that
$$\begin{cases} \text{i) } \left\| z'\left(A-\lambda_0\right)\left(A-z\right)^{-1}\left[\left(A-\lambda_0\right)^{-1};\left(B+z'\right)^{-1}\right]\right\|_{\mathcal{L}(X)} \leq \\ \qquad \leq \psi\left(|z|,|z'|\right), \forall z \in \Sigma_A, \forall z' \in \Sigma_{-B} \\ \text{ii) } \displaystyle\int_{\gamma_\lambda} \psi(|z+\lambda|,|z|)|dz| \underset{\lambda\to+\infty}{\longrightarrow} 0. \end{cases} \tag{2.7}$$

Hypothesis (H.3): A and B satisfy (H.2) where ψ satisfies the following conditions:
$$\exists C > 0, \quad h \in \mathbb{N}; \quad (\alpha_i)_{i=1,\dots,h} \quad \text{and} \quad (\beta_i)_{i=1,\dots,h} \subset \mathbb{R}$$
such that
$$\begin{cases} \text{i) } \psi\left(|z|,|z'|\right) < C \displaystyle\sum_1^n \frac{1}{|z|^{\alpha_i}\left|z'\right|^{\beta_i}}, \quad \forall z \in \Sigma_A, \forall z' \in \Sigma_{-B}, \\ \text{ii) } 0 \leqslant 1-\alpha_i < \beta_i \leqslant 2, \quad \forall i = 1,\dots,h. \end{cases} \tag{2.8}$$
We will set
$$\delta = \min_i \left(\alpha_i + \beta_i - 1\right). \tag{2.9}$$

Remark 2.2.1. If ψ verifies conditions i) and ii) of (H.3) then condition ii) of hypothesis (H.2) is verified. Indeed,
$$\int_{\gamma_\lambda} \psi(|z+\lambda|,|z|)|dz| \leqslant C \int_{\gamma_\lambda} \sum_1^h \frac{1}{|z+\lambda|^{\beta_i}} \frac{1}{|z|^{\alpha_i}}|dz| \leqslant$$
$$\leqslant C \sum_1^h \int_{\gamma_\lambda} \frac{1}{|\mu\lambda+\lambda|^{\beta_i}} \frac{1}{|\mu\lambda|^{\alpha_i}} \lambda|d\mu| = 0\left(\frac{1}{\lambda^\delta}\right).$$

We have the following two technical lemmas:

Lemma 2.2.1. *$\forall \mu, z \in \rho_A$ and $\forall z' \in \rho_{-B}$ we have*

- $\left[(A-z)^{-1};(B+z')^{-1}\right] = (A-\mu)(A-z)^{-1}\left[(A-\mu)^{-1};(B+z')^{-1}\right](A-\mu)(A-z)^{-1}$.
- $\left[A(A-z)^{-1};(B+z')^{-1}\right] = z\left[(A-z)^{-1};(B+z')^{-1}\right]$.
- $\left[(A-z)^{-1};B\,(B+z')^{-1}\right] = z'\left[(A-z)^{-1};(B+z')^{-1}\right]$.

Proof. It suffices to develop the commutators. □

Lemma 2.2.2. *If A and B satisfy (H.2) then we have*

$$\left\| z'(A-\lambda)(A-z)^{-1}\left[(A-\lambda)^{-1};(B+z')^{-1}\right]\right\|_{\mathcal{L}(X)} \leqslant$$
$$\leqslant \psi\left(|z|,|z'|\right)\left(1+\frac{C_A\cdot|\lambda-\lambda_0|}{\lambda}\right), \quad \forall\lambda>0, \forall z\in\Sigma_A, \forall z'\in\Sigma_{-B}.$$

Lemma 2.2.3. *If A and B satisfy (H.2) then $\forall \lambda > \lambda_0, \exists \varphi_\lambda :]\,0, +\infty\,[\to \mathbb{R}_+$ such that*

i) $\displaystyle\int_{\gamma_\lambda} \varphi_\lambda(|\ z\ |)|dz| < +\infty, \forall\lambda > \lambda_0.$

ii) $\left\|\left[(A-\lambda)^{-1};(B+z)^{-1}\right]\right\|_{\mathcal{L}(X)} \leq \varphi_\lambda(|z|), \forall z\in\Sigma_{-B}.$

2.2.3 Representation of the solution

Under the previous assumptions, we want to obtain a formula representing the possible solution x of (2.6) using the operator S_λ and the regularity of the second member y.

Proposition 2.2.1. *It is assumed that hypotheses (H.1) and (H.2) are verified. For $y \in X, \lambda > 0$, let $x \in D_A \cap D_B$ be a solution of* (2.6) *then*

$$x + I_\lambda(x) = S_\lambda y \tag{2.10}$$

where

$$I_\lambda(x) = -\frac{1}{2\pi i}\int_{\gamma_\lambda} z(A-\lambda-z)^{-1}\left[(A-\lambda)^{-1};(B+z)^{-1}\right](A-\lambda)x dz.$$

Proposition 2.2.2. *Under hypotheses (H.1) and (H.2), we have*

i) if $y \in D_A(\sigma, +\infty)$ then $S_\lambda y \in D_A$ and

$$(A-\lambda)S_\lambda y = -\frac{1}{2\pi i}\int_{\gamma_\lambda} z\left[(A-\lambda-z)^{-1};(B+z)^{-1}\right] ydz-$$

$$-\frac{1}{2\pi i}\int_{\gamma_\lambda}(B+z)^{-1}(A-\lambda)(A-\lambda-z)^{-1}ydz,$$

ii) if $y \in D_B(\sigma, +\infty)$ then $S_\lambda y \in D_A$ and

$$(A-\lambda)S_\lambda y = \frac{1}{2\pi i}\int_{\gamma_\lambda}(A-\lambda-z)^{-1}B(B+z)^{-1}ydz + y.$$

Thanks to the hypothesis (H.2) and for $y \in D_A(\sigma, +\infty)$ (or $D_B(\sigma, +\infty)$) we can apply $(A-\lambda)$ to both members of (2.10). More precisely, we have

Proposition 2.2.3. *We assume that (H.2) holds and that $y \in D_A(\sigma, +\infty)$ (or $y \in D_B(\sigma, +\infty)$); then if x is a solution of* (2.6) *we have*

$$(A-\lambda)x + J_\lambda(A-\lambda)x = (A-\lambda)S_\lambda y, \tag{2.11}$$

where

$$J_\lambda(v) = -\frac{1}{2\pi i}\int_{\gamma_\lambda} z(A-\lambda)(A-\lambda-z)^{-1}\left[(A-\lambda)^{-1};(B+z)^{-1}\right]vdz.$$

Proof. It is enough to notice that (H.2) allows to give a meaning to $(A-\lambda)I_\lambda x$ and proposition 2.2.2 allows to make $(A-\lambda)S_\lambda y$. □

Equation (2.11) has the form $v + J_\lambda(v) = h$, h given; so we need to invert the continuous operator $(1 + J_\lambda)$ for some λ.

Proposition 2.2.4. *We assume that (H.2) holds; then there exists $\bar{\lambda} > 0$ such that $\forall \lambda \geqslant \bar{\lambda}$ the operator $(1 + J_\lambda)$ is invertible and $(1 + J_\lambda)^{-1} \in \mathcal{L}(X)$, additionally if $y \in D_A(\sigma, +\infty)$ (or $y \in D_B(\sigma, +\infty)$) then*

$$x = (A-\lambda)^{-1}(1 + J_\lambda)^{-1}(A-\lambda)y. \tag{2.12}$$

Proof. Thanks to ii) of the hypothesis (H.2), $\exists\bar{\lambda} > 0$ such that

$$\int_{\gamma_\lambda} \psi(|z+\lambda|, |z|)|dz| \leq \frac{1}{2}, \quad \forall\lambda \geq \bar{\lambda},$$

and therefore $\|J_\lambda\|_{\mathcal{L}(X)} \leq \frac{1}{2}, \forall\lambda > \bar{\lambda}$, hence the invertibility and continuity of $(1+J_\lambda)^{-1}$. Applying $(1+J_\lambda)^{-1}$ to (2.4), we get (2.12). □

Corollary 2.2.1. *(Uniqueness) We assume the hypotheses of proposition 2.2.4; then if x is the solution to problem* (2.6)*, x is unique.*

Proof. Indeed if x_1 and x_2 are two solutions of (2.6), setting $x = x_1 - x_2$ we have $Ax + Bx - \lambda x = 0$ and according to the proposition 2.2.4 we have

$$x = x_1 - x_2 = (A-\lambda)^{-1}\left\{(1+J_\lambda)^{-1}(A-\lambda)(0)\right\} = 0.$$

□

2.2.4 Approached problem

In summary of the preceding paragraphs, we have shown, thanks essentially to the regularity of the right hand side y and to the hypothesis (H.2), that the existence of a solution x of (2.6) implies its representation by (2.12) for $\lambda > \bar{\lambda}$ and hence uniqueness. To show the existence of $x \in D_A \cap D_B$ solution of (2.6) we will consider the sequence (x_n) of solutions of the approximate problems of (2.6) using Yosida approximants of A defined by $A_n = nA(n-A)^{-1}, n \in \mathbb{N}^*$. Therefore, we consider the problems:

$$(L_n - \lambda x) = (A_n + B - \lambda)x = y. \tag{2.13}$$

We will prove the following proposition:

Proposition 2.2.5. *There exists $\lambda^* > 0$ such that for each $n \geqslant 1$, $\lambda > \lambda^*$ and $y \in X, \exists! x_n \in D_B$ solution of* (2.13)*; in addition we have*

$$x_n = (A_n - \lambda)^{-1}(1+J_{n,\lambda})^{-1}(A_n-\lambda)\, S_{n,\lambda}y,$$

where

$$\begin{cases} J_{n,\lambda}(v) = \frac{1}{2\pi i}\int_{\gamma_\lambda} z\,(A_n-\lambda)\,(A_n-\lambda-z)^{-1}\left[(A_n-\lambda)^{-1}; (B+z)^{-1}\right] v dz \\ S_{n,\lambda}y = -\frac{1}{2\pi i}\int_{\gamma_\lambda}(A_n-\lambda-z)^{-1}(B+z)^{-1} y dz. \end{cases} \tag{2.14}$$

For the proof of this proposition we need several lemmas.

Lemma 2.2.4. *If A and B satisfy (H.2) then $\exists M\left(\theta_A,\lambda_0\right), N\left(B,A,\theta_0\right)$ such that*

$$\left\{\begin{array}{l} i)\ \left\| z'\left(A_n-\lambda\right)\left(A_n-z\right)^{-1}\left[\left(A_n-\lambda\right)^{-1};\left(B+z'\right)^{-1}\right]\right\|_{\mathcal{L}(X)} \leq \\ \leq M\left(\theta_A,\lambda_0\right)\psi\left(|z|,|z'|\right), \forall n\geq 0, \forall\lambda\geq\lambda_0, \forall z\in\Sigma_A, \forall z'\in\Sigma_{-B} \\ ii)\ \left\|S_{n,\lambda}y\right\|_X \leq \dfrac{N\left(A,B,\theta_0\right)}{\lambda}\|y\|_X, \quad \forall y\in X. \end{array}\right. \tag{2.15}$$

Lemma 2.2.5. *For each $n>0$ there exists $\lambda(n)>0$ such that*

$$[\lambda(n),+\infty[\subset\rho(L_n)\ (\ i.e.\ \forall\lambda\geq\lambda(n),(L_n-\lambda)^{-1}\in\mathcal{L}(X)).$$

Lemma 2.2.6. *There exists $\lambda^*>0$ such that for each $n\geqslant 1,\lambda>\lambda^*$; $\exists M(n)$ such that*

$$\|x\|_X\leqslant M(n)\frac{\left\|\left(L_n-\lambda\right)x\right\|_X}{|\lambda|}, \quad \forall x\in D\left(L_n\right)=D_B. \tag{2.16}$$

We will now study the convergence of the approximate problems $\left(L_n-\lambda\right)x=y$; for this we look at the limits of $\left(A_n-\lambda\right)S_{n,\lambda}y$ and operators $J_{n,\lambda}$ defined in (2.14).

Proposition 2.2.6. *For each set $\lambda\geqslant\lambda^*$ (see Lemma 2.2.6) and $y\in D_A(\sigma,+\infty)$ (or $y\in D_B(\sigma,+\infty)$) then $\left(A_n-\lambda\right)S_{n,\lambda}y\underset{n\to+\infty}{\longrightarrow}(A-\lambda)S_\lambda y$.*

Proposition 2.2.7. *Suppose (H.1), (H.2) and let λ whatever such that $\lambda\geq\lambda^*$, then we have*

i) $\forall n>0, \|J_{n,\lambda}\|\leqslant\dfrac{1}{2}$.

ii) $\forall x\in X, \|J_{n,\lambda}x-J_\lambda x\|_X\underset{n\to+\infty}{\longrightarrow}0$.

iii) $\forall n>0, (1+J_{n,\lambda})^{-1}$ *exists and* $||(1+J_{n,\lambda})^{-1}||_{\mathcal{L}(X)}\leqslant 2$, *in addition*

$\forall x\in X, \left\|\left(1+J_{n,\lambda}\right)^{-1}x-\left(1+J_\lambda\right)^{-1}x\right\|_X\underset{n\to+\infty}{\longrightarrow}0.$

2.2.5 Strict solution

We are now ready to state and prove the existence of the solution to the problem (2.6).

Theorem 2.2.1. *Assume that A and B verify (H.1), (H.2); then there exists $\lambda^*>0$ such that $\forall\lambda\geqslant\lambda^*$ and $\forall y\in D_A(\sigma,+\infty)$ (or $y\in D_B(\sigma,+\infty)$) the equation $Ax+Bx-\lambda x=y$ admits a unique solution $x\in D(L)=D_A\cap D_B$.*

Proof. Let λ^* be the number defined in lemma 2.2.6 and $\lambda \geq \lambda^*$; let $y \in D_A(\sigma, +\infty)$ (or $y \in D_B(\sigma, +\infty)$), then consider the vector x defined by

$$x = (A - \lambda)^{-1} (1 + J_\lambda)^{-1} (A - \lambda) S_\lambda y \in D_A, \tag{2.17}$$

where J_λ and S_λ are defined in (2.11) and (2.5) respectively, and consider also the vector x_n defined by

$$x_n = \big(A_n - \lambda\big)^{-1} (1 + J_{n,\lambda})^{-1} (A_n - \lambda)\, S_{n,\lambda} y, \tag{2.18}$$

where $J_{n,\lambda}$ and $S_{n,\lambda}$ are given in (2.14); $x_n \in D_B$ is solution of the problem $(A_n + B - \lambda)\, x_n = y$. By proposition 2.2.7 we have $x_n \underset{n \to +\infty}{\longrightarrow} x$ in E; but

$$Bx_n = -\,(A_n - \lambda)\, x_n + y = -\,(1 + J_{n,\lambda})^{-1} (A_n - \lambda)\, S_{n,\lambda} y + y,$$

so

$$Bx_n \to -\,(1 + J_\lambda)^{-1} (A - \lambda) S_\lambda y + y = -(A - \lambda)x + y,$$

and as B is closed we have

$$x \in D_B \cap D_A \quad \text{ and } \quad Bx = -(A - \lambda)x + y,$$

($x \in D_A$ by its representation). $\square$

We arrive in the previous paragraphs at the existence and uniqueness of the solution x of (2.6) in the case where y is in $D_A(\sigma, +\infty)$ or $D_B(\sigma, +\infty)$; it is interesting to do it also for y in $D_A(\sigma, p)$ or $D_B(\sigma; p)$ for $1 \leqslant p < +\infty$. Thanks to the inclusions already mentioned:

$$D_A(\sigma, p) \hookrightarrow D_A(\sigma, +\infty), \quad \forall p \in [1, +\infty[,$$

we deduce the corollary:

Corollary 2.2.2. *Suppose that A and B satisfy (H.1), (H.2), then there exists $\lambda^* > 0$ such that $\forall \lambda \geq \lambda^*$ and $\forall y \in D_A(\sigma, p)(1 \leqslant p < +\infty)$ (or $y \in D_B(\sigma, p)$) the equation $Ax + Bx - \lambda x = y$ admits a unique solution $x \in D_A \cap D_B$.*

2.2.6 Regularity of the solution

To study the regularity of the strict solution of problem (2.6), the hypotheses (H.1), (H.2) are not sufficient because the function $\psi(|z+\lambda|,|z|)$ is not explicit and the regularity of (2.6) depends in some way on the "homogeneity" of ψ. We will then need the hitherto unused hypothesis (H.3) which explains ψ.

Now we will cite three useful lemmas for the sequel. We then assume that (H.1), (H.2) and (H.3) hold and that the number δ is defined in (2.9). So we have

Lemma 2.2.7. *i)* $(A-\lambda)S_\lambda \in \mathcal{L}\left(D_A(\theta,+\infty)\right), \forall\theta \in]\,0,1[$,

ii) $(A-\lambda)S_\lambda \in \mathcal{L}\left(D_A(\theta,+\infty); D_B(\sigma,+\infty)\right), \forall\theta \in]\,0,1[$ *and* $\forall\sigma \in\, \in]0,\beta\wedge\theta]$ *where* $\beta\wedge\theta = \min(\beta,\theta)$ *and* $\beta = \min\{\beta_i, i=1,\dots,h\}$.

Lemma 2.2.8. *i)* $(A-\lambda)S_A \in \mathcal{L}\left(D_B(\theta,+\infty); D_B(\sigma,+\infty)\right), \forall\theta \in]\,0,1[$ *and* $\forall\sigma \in]0,\beta\wedge\theta]$,

ii) $[(A-\lambda)S_\lambda - I] \in \mathcal{L}\left(D_B(\theta,+\infty); D_A(\theta,+\infty)\right), \forall\theta \in]\,0.1[$.

Lemma 2.2.9. *i)* $J_\lambda \in \mathcal{L}\left(E, D_A(\sigma,+\infty)\right), \forall\sigma \in]\,0,\delta] \cap]\,0,1[$,

ii) $J_\lambda \in \mathcal{L}\left(E; D_B(\sigma,+\infty)\right), \forall\sigma \in]\,0,\delta[\cap]0,1[$,

iii) $J_\lambda \in \mathcal{L}\left(D_B(\varepsilon,+\infty), D_B(\sigma,+\infty)\right), \forall\sigma \in]\,0,\delta] \cap]\,0,1[\ \forall\varepsilon \in]0,1[$.

Theorem 2.2.2. *Suppose (H.1), (H.2), (H.3) and y in $D_A(\theta,+\infty)$; let then x be the unique solution of* (2.6). *So we have*

i) $(A-\lambda)x \in D_A(\sigma,+\infty),\ \forall\sigma \leqslant \delta\wedge\theta$,

ii) $Bx \in D_A(\sigma,+\infty),\qquad \forall\sigma \leqslant \delta\wedge\theta$,

iii) $(A-\lambda)x \in D_B(\sigma,+\infty), \forall\sigma \leqslant \delta\wedge\theta$.

Proof. We have $(A-\lambda)x = -J_\lambda((A-\lambda)x) + (A-\lambda)S_\lambda y$ and therefore by applying lemma 2.2.7 and the lemma 2.2.9 we obtain $(A-\lambda)S_\lambda y \in D_A(\theta,+\infty)$ and $-J_\lambda((A-\lambda)x) \in D_A(\sigma,+\infty), \forall\sigma \in]\,0,\delta] \cap]0.1[$ hence point i). Point iii) is obtained in the same way. As for ii) it suffices to use the equation $Bx + (A-\lambda)x = y$. □

Theorem 2.2.3. *We always assume (H.1), (H.2), (H.3), and $y \in D_B(\theta,+\infty)$ and let x be the solution of (2.6); then* *i)* $(A-\lambda)x \in D_B(\sigma,+\infty), \forall\sigma \leqslant \delta\wedge\theta,$

ii) $Bx \in D_B(\sigma,+\infty), \qquad \forall\sigma \leqslant \delta\wedge\theta,$

iii) $Bx \in D_A(\sigma,+\infty), \qquad \forall\sigma \leqslant \delta\wedge\theta.$

Proof. We still use $(A-\lambda)x = -J_\lambda((A-\lambda)x) + (A-\lambda)S_\lambda y$ and the lemmas 2.2.8 and 2.2.9 i). We have $(A-\lambda)x \in D_B(\sigma,+\infty)$, $\forall\sigma < \delta\wedge 1$, and from lemma 2.2.9 iii) we deduce that $(A-\lambda)x \in D_B(\sigma,+\infty)$, $\forall\sigma \in]0,\delta]\cap]0,1[$, and so i) and ii) thanks to the equation. On the other hand $Bx = -(A-\lambda)x + y = J_\lambda((A-\lambda)x) - [(A-\lambda)S_\lambda - I]\, y$; Lemmas 2.2.8 ii) and 2.2.9 i) conclude for point iii) of the theorem.

□

If $y \in D_A(\theta,p)$ (or $D_B(\theta,p), 1 \leqslant p < +\infty$), we do not obtain the maximal regularity $\delta\wedge\theta$ as in theorem 2.2.2 or 2.2.3; more precisely we have

Theorem 2.2.4. *We suppose (H.1), (H.2), (H.3) satisfied and let* $y \in D_A(\theta,p)(\theta \in]0,1[; 1 \le p < +\infty)$, *then the unique solution* x *of* (2.6) *for* $\lambda \geqslant \lambda^*$ *verifies*

i) $(A-\lambda)x \in D_A(\sigma,p), \quad \forall\sigma \in]\,0,\delta[\cap]0,\theta],$

ii) $Bx \in D_A(\sigma,p), \qquad \forall\sigma \in]\,0,\delta[\cap]0,\theta],$

iii) $(A-\lambda)x \in D_B(\sigma,p), \forall\sigma \in]\,0,\delta[\cap]0,\theta].$

We have an analogous statement if $y \in D_B(\sigma,p)$.

Proof. We will only demonstrate point i), the proof of points ii) and iii) being analogous

1^{st} Case. $\delta \in]0.1\,[$ and $\theta \geqslant \delta$; let $y \in D_A(\theta,p)$, then $y \in D_A(\theta,+\infty)$. So for $\lambda \geqslant \lambda^*, \exists! x \in D_A\cap D_B$ such that $Ax+Bx-\lambda x = y$ and Theorem 2.2.2 implies that $(A-\lambda)x \in D_A(\delta,+\infty)$.

But $D_A(\delta,+\infty) \subset D_A(\sigma,p)$, $\forall\sigma < \delta$ and $\forall p \in [1,+\infty[$, so $(A-\lambda)x \in D_A(\sigma,p)$, $\forall\sigma < \delta$.

2^{nd} Case. $\delta > \theta$; let θ_1 and θ_2 in $]0,1\,[$ such that $\theta_1 < \theta < \theta_2$ and $\theta_2 < \delta$; let T be the linear map defined on $D_A(\omega,+\infty)$ where $\omega < \theta_1$ by

$$T : D_A(\omega,+\infty) \to X$$

$$y \to T(y) = (A-\lambda)x,$$

x being the solution of $Ax+Bx-\lambda x = y$.

Theorem 2.2.2 then expresses that the restriction of T to the spaces $D_A(\theta_i, +\infty)$ is continuous linear on themselves and therefore by applying the fundamental theorem of interpolation [33]; $T \in \mathcal{L}\left((D_A(\theta_2, +\infty); D_A(\theta_1, +\infty))_{s,q}\right)$ and this $\forall s \in]0,1[, \forall q \in [1, +\infty[$. In particular if we take for $s = (\theta_2 - \theta)/(\theta_2 - \theta_1)$ and $q = p$ we obtain of the iteration theorem

$$(D_A(\theta_2, +\infty); D_A(\theta_1, +\infty))_{(\theta_2-\theta)/(\theta_2-\theta_1),p} = D_A(\theta, p),$$

hence the result. $\square$

Chapter 3

L^p-Regularity results for 2m-th order parabolic equations

This chapter is devoted to the analysis of the following linear $2m$-th order parabolic equation

$$\partial_t u + (-1)^m \sum_{k=1}^{N} \partial_{x_k}^{2m} u = f,$$

subject to Dirichlet type condition

$$\partial_\nu^l u = 0,\ l = 0, 1, ..., m-1,$$

on the lateral boundary, where m is a positive integer. The right-hand side f of the equation is taken in the Lebesgue space $L^p,\ 1 < p < +\infty$. The problem is set in a domain of the form

$$Q = \left\{ (t, x_1, ..., x_N) \in \mathbb{R}^{N+1} : 0 \leq \sqrt{x_1^2 + ... + x_N^2} < t^\alpha,\ 0 < t < 1 \right\},$$

with $\alpha > 1/2m$.

3.1 Introduction

Let Q be an open set of $\mathbb{R}^{N+1}$ defined by

$$Q = \left\{ (t, x_1, x_2, ..., x_N) \in \mathbb{R}^{N+1} : (x_1, x_2, ..., x_N) \in \Omega_t,\ 0 < t < 1 \right\},$$

where for a fixed t in the interval $]0,1[$, Ω_t is a bounded domain of $\mathbb{R}^N$, $N>1$, defined by

$$\Omega_t=\left\{(x_1,x_2,...,x_N)\in\mathbb{R}^N : 0\leq\sqrt{x_1^2+x_2^2+...+x_N^2}<t^\alpha\right\},$$

with $\alpha>1/2m$ and m belongs to the set of all nonzero natural numbers $\mathbb{N}^*$. In Q, consider the boundary value problem

$$\begin{cases}\partial_t u+Mu=f\in L^p\left(Q\right),\\ \partial_\nu^l u\big|_{\partial Q\smallsetminus\Gamma_1}=0,\ l=0,1,...,m-1,\end{cases}\tag{3.1}$$

where $M=(-1)^m\sum_{k=1}^N\partial_{x_k}^{2m}$, ∂Q is the boundary of Q, Γ_1 is the part of the boundary of Q where $t=1$ and ∂_ν^l stands for the derivative of order l throughout the normal vector ν on $(\partial Q\smallsetminus\Gamma_1)$. Here, $L^p(Q)$, $1<p<+\infty$, denotes the space of p-integrable functions on Q with the measure $dtdx_1...dx_N$.

If the domain under investigation is a cylinder, the solvability of the corresponding problem is known over the scales of anisotropic Sobolev-Slobodetskii or Hölder spaces since the mid of the last century. Indeed, classical results on the resolution of Problem (3.1) can be found in [22] and [23] and in the references therein. Some recent regularity results are given in [2], [8], [11], [12], [16], [27], [28] and [29].

Besides being interesting in itself, the study of Problem (3.1) is motivated by the interest of researchers for many mathematical questions related to non-regular domains. During the last decades and since many applied problems lead directly to boundary-value problems in "bad" domains, numerous authors studied partial differential equations in non-smooth domains. Among these which are related to higher order parabolic equations we can cite Baderko [1], Cherepova [3], Cherfaoui *et al.* [5], Galaktionov [13], Grimaldi [14], Mikhailov [25], [26], Sadallah [31] and the references therein.

The L^2-solvability of Problem (3.1) has been investigated in [6] by the domain decomposition method, see also [4] and [17]. The difficulty with the space L^p, $p\neq 2$, is that this space is not a Hilbert space. So, the domain decomposition method used in [6] does not seem to be appropriate for our study and cannot be generalized in this sense. An idea for this extension (to the case L^p, $p\in(1,\infty)$) can be found in [8] and [18], in which the operators' sum method was used. This method is interesting because it may be generalized to Banach spaces instead of Hilbert spaces. For more details and recent results concerning this method, see [7], [30] and the references therein.

In this work, we are especially interested in the question of what sufficient conditions, as weak as possible, the dimension N, the exponent p and the type of the domain Q must verify in order that Problem (3.1) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_{0,p}^{1,2m}(Q) := \left\{ u \in H_p^{1,2m}(Q) : \partial_\nu^l u\big|_{\partial Q \setminus \Gamma_1} = 0,\ l = 0, 1, ..., m-1 \right\},$$

with

$$H_p^{1,2m}(Q) = \{u : \partial_t u, \partial^\alpha u \in L^p(Q), |\alpha| \leq 2m\},$$

where $\alpha = (i_1, i_2, ..., i_N) \in \mathbb{N}^N$, $|\alpha| = i_1 + i_2 + ... + i_N$ and $\partial^\alpha u = \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} ... \partial_{x_N}^{i_N} u$. The space $H_p^{1,2m}(Q)$ is equipped with the natural norm, that is

$$\|u\|_{H_p^{1,2m}(Q)} = \left(\|\partial_t u\|_{L^p(Q)}^p + \sum_{|\alpha| \leq 2m} \|\partial^\alpha u\|_{L^p(Q)}^p \right)^{1/p}.$$

The main assumption is

$$\frac{1}{2m} < \alpha < \frac{p-1}{N}. \tag{3.2}$$

The outline of this chapter is as follows. In Section 3.2 we recall the essential of the sum's theory we will have to apply. In Section 3.3 we perform a change of variables conserving (modulo a weight) the spaces L^p and $H_p^{1,2m}$, and transforming Problem (3.1) into a degenerate parabolic problems in a cylindrical domain. Section 3.4 is concerned with the application of the sum of operators' method to the transformed problem. Finally, in Section 3.5 we give results concerning the transformed problem and we return to our initial problem by using the inverse change of variables.

3.2 On the non-commutative sum of linear operators

Let Λ be a closed linear operator in a complex Banach space E. Then, Λ is said to be sectorial if

(i) $D(\Lambda)$ and $Im(\Lambda)$ are dense in E,

(ii) $\ker(\Lambda)=\{0\}$,

(iii) $]-\infty,0[\subset\rho(\Lambda)$ ($\rho(\Lambda)$ is the resolvent set of Λ) and there exists a constant $K\geq 1$ such that $\forall t>0,\ \left\|t(\Lambda+tI)^{-1}\right\|_{L(E)}\leq K$. If Λ is sectorial it follows easily that $\rho(-\Lambda)$ contains an open sector $\Sigma_{\varphi}:=\{z\in\mathbb{C}:z\neq 0,|\arg z|<\varphi\}$, with $\varphi\in\,]0,\pi[$.

Consider two closed linear operators A and B with dense domains $D(A)$ and $D(B)$ respectively in E. Assume that both operators satisfy the following assumptions of Da Prato-Grisvard's type [7].

There exist positive numbers r, C_A, C_B, θ_A, θ_B such that

$$\theta_A+\theta_B<\pi, \tag{3.3}$$

$$\begin{array}{l}\rho(-A)\supset\Sigma_{\pi-\theta_A}:=\{z\in\mathbb{C}:|z|\geq r,|\arg z|<\pi-\theta_A\}\\ \text{and }\forall\lambda\in\Sigma_{\pi-\theta_A},\ \left\|(A+\lambda I)^{-1}\right\|_{L(E)}\leq\dfrac{C_A}{|\lambda|},\end{array} \tag{3.4}$$

$$\begin{array}{l}\rho(-B)\supset\Sigma_{\pi-\theta_B}:=\{z\in\mathbb{C}:|z|\geq r,|\arg z|<\pi-\theta_B\}\\ \text{and }\forall\mu\in\Sigma_{\pi-\theta_B},\ \left\|(B+\mu I)^{-1}\right\|_{L(E)}\leq\dfrac{C_B}{|\lambda|}.\end{array} \tag{3.5}$$

We also assume that there are constants $C>0$, $\lambda_0>0$, (with $\lambda_0\in\rho(-A)$), τ and ρ such that

$$\left\{\begin{array}{l}(i)\ \left\|(A+\lambda_0 I)(A+\lambda I)^{-1}\left[(A+\lambda_0 I)^{-1};(B+\mu I)^{-1}\right]\right\|_{L(E)}\\ \leq\dfrac{C}{|\lambda|^{1-\tau}.|\mu|^{1+\rho}}\ \forall\lambda\in\rho(-A),\ \forall\mu\in\rho(-B),\\ \\ (ii)\ 0\leq\tau<\rho\leq 1.\end{array}\right. \tag{3.6}$$

For more details concerning this last Labbas-Terreni commutator assumption, see [19] and [20].

For any $\sigma\in\,]0,1[$ and $1\leq p\leq+\infty$, let us introduce the real Banach interpolation spaces $D_A(\sigma,p)$ between $D(A)$ and E (or $D_B(\sigma,p)$ between $D(B)$ and E) which are characterized (for $1\leq p<+\infty$) by

$$D_A(\sigma,p)=\left\{\xi\in E:t\longmapsto\left\|t^{\sigma}A(A+tI)^{-1}\xi\right\|_E\in L_*^p\right\},$$

where L_*^p denotes the space of p-integrable functions on $(0,+\infty)$ with the measure dt/t. For $p=+\infty$,

$$D_A(\sigma,+\infty)=\left\{\xi\in E:\sup_{t>0}\left\|t^\sigma A(A+tI)^{-1}\xi\right\|_E<\infty\right\}.$$

For these spaces, see [15]. Then the main result proved in Labbas-Terreni [19] is

Theorem 3.2.1. *Under assumptions (3.3), (3.4), (3.5) and (3.6), there exists $\lambda^*>0$ such that for any $\lambda\geq\lambda^*$ and for any $h\in D_A(\sigma,p)$, equation*

$$Aw+Bw+\lambda w=h,$$

has a unique solution $w\in D(A)\cap D(B)$ with the regularities Aw, $Bw\in D_A(\theta,p)$ and $Aw\in D_B(\theta,p)$ for any θ verifying $\theta\leq\min(\sigma,\rho-\tau)$.

3.3 Change of variables and operational setting of the problem

3.3.1 Change of variables

We make the following change of variables and functions

$$\begin{array}{rcl}\Pi\ :Q & \longrightarrow & G\\ (t,x_1,x_2,...,x_N) & \longmapsto & (t,y_1,y_2,...,y_N)=\left(t,\dfrac{x_1}{t^\alpha},\dfrac{x_2}{t^\alpha},...,\dfrac{x_N}{t^\alpha}\right),\end{array}$$

where $G=\left]0,1\right[\times B(0,1)$, with $B(0,1)$ is the unit ball of $\mathbb{R}^N$ centered at the origin. Set $u(t,x_1,x_2,...,x_N)=v(t,y_1,y_2,...,y_N)$ and $f(t,x_1,x_2,...,x_N)=g(t,y_1,y_2,...,y_N)$, then Problem (3.1) is transformed, in G, into the following degenerate evolution problem

$$\begin{cases} t^{2m\alpha}\partial_t v+Mv-\alpha t^{2m\alpha-1}\sum_{k=1}^N y_k\partial_{y_k}v=t^{2m\alpha}g=h,\\ v|_{\Sigma_0}=0,\\ \partial_\nu^l v|_{\partial G\setminus(\Sigma_0\cup\Sigma_1)}=0,\ l=0,1...,m-1,\end{cases}\tag{3.7}$$

where $M=\sum_{k=1}^{N}\partial_{y_k}^{2m}$, $\Sigma_j,\ j=0,\ 1$ is the part of the boundary of G where $t=j$. It is easy to see that $f\in L^p(Q)$ if and only if $t^{N\alpha/p}g\in L^p(G)$. Indeed,

$$\begin{aligned} f\in L^p(Q) &\Leftrightarrow \int_0^1\int_{\Omega_t}|f(t,x_1,...,x_N)|^p\,dt\,dx_1...dx_N &<& +\infty \\ &\Leftrightarrow \int_0^1\int_{B(0,1)}\left|t^{N\alpha/p}g(t,y_1,...,y_N)\right|^p dt\,dy_1...dy_N &<& +\infty \\ &\Leftrightarrow t^{N\alpha/p}g\in L^p(G). \end{aligned}$$

Consequently, $f\in L^p(Q)$ if and only if $t^{-2m\alpha+(N\alpha/p)}h\in L^p(G)$ which implies that $h\in L^p(G)$, since $h=\left(t^{-2m\alpha+(N\alpha/p)}h\right)t^{2m\alpha-(N\alpha/p)}$ and $2m\alpha-(N\alpha/p)>0$. Then, the function $h=t^{2m\alpha}g$ lies in the closed subspace of $L^p(G)$ defined by

$$E=\left\{h\in L^p(0,1;L^p(B(0,1))):t^{-2m\alpha+(N\alpha/p)}h\in L^p(0,1;L^p(B(0,1)))\right\}.$$

This space is equipped with the norm $\|h\|_E=\left\|t^{-2m\alpha+(N\alpha/p)}h\right\|_{L^p(0,1;L^p(B(0,1)))}$.

3.3.2 Operational formulation of Problem (3.7)

Recall that $\alpha>1/2m$ and assume

$$p>1+N\alpha. \tag{3.8}$$

Set $X=L^p(B(0,1))$ and define the functions

$$\begin{aligned} v\ &:[0,1]\longrightarrow X;\quad t\longmapsto v(t);\quad v(t)(y_1,y_2,...,y_N)=v(t,y_1,y_2,...,y_N), \\ h\ &:[0,1]\longrightarrow X;\quad t\longmapsto h(t);\quad h(t)(y_1,y_2,...,y_N)=h(t,y_1,y_2,...,y_N). \end{aligned}$$

Consider the family of operators $(L(t))_{t\in[0,1]}$ defined by

$$\begin{cases} D(L(t)) &= \left\{\psi\in W^{2m,p}(B(0,1)):\partial_\nu^l\psi\big|_{\partial B(0,1)}=0,\ l=1,...,m-1\right\}, \\ (L(t)\psi) &= M\psi-\alpha t^{2m\alpha-1}\sum_{k=1}^{N}y_k\partial_{y_k}\psi \text{ for a.e. } t\in(0,1), \end{cases}$$

then Problem (3.7) is equivalent to the following operational degenerate Cauchy problem in X

$$\begin{cases} t^{2m\alpha}v'(t)+L(t)v(t) &= h(t),\quad t\in(0,1), \\ v(0)=0. \end{cases} \tag{3.9}$$

Observe that $\overline{D(L(t))} = X$. Set

$$\begin{cases} w(t) = e^{\lambda \frac{t^{1-2m\alpha}}{1-2m\alpha}} v(t), \\ k(t) = e^{\lambda \frac{t^{1-2m\alpha}}{1-2m\alpha}} h(t), \end{cases}$$

where λ is some positive number. Then, w verifies

$$\begin{cases} t^{2m\alpha} w'(t) + L(t) w(t) + \lambda w(t) = k(t), \quad t \in (0,1), \\ w(0) = 0, \end{cases} \tag{3.10}$$

where k belongs to the space

$$E = \left\{ h \in L^p(0,1;X) : t^{-2m\alpha + (N\alpha/p)} h \in L^p(0,1;X) \right\}.$$

We obtain then the new operational form of the previous problem, mainly

$$Aw + Bw + \lambda w = k,$$

where

$$(Aw)(t) = L(t) w(t), \ t \in [0,1],$$

with domain

$$D(A) = \left\{ w \in E : t^{-2m\alpha + (N\alpha/p)} w \in L^p\left(0,1; W^{2m,p}(B(0,1)) \cap W_0^{m,p}(B(0,1))\right) \right\},$$

and

$$(Bw)(t) = t^{2m\alpha}(t) w'(t), \ t \in [0,1],$$

with domain

$$D(B) = \left\{ w \in E : t^{(N\alpha/p)} w' \in L^p(0,1;X) \text{ and } w(0) = 0 \right\}.$$

Note that the trace $w(0)$ is well defined in $D(B)$. In fact, we have

$$t^{N\alpha/p} w \in L^p(0,1;X), \ t^{N\alpha/p} w' \in L^p(0,1;X),$$

and in virtue of (3.8) $(N\alpha/p) + (1/p) < 1$. Then, w is continuous on $[0,1]$, (see [33, Lemma, p. 42]).

3.4 Application of the sums

Now we are in position to apply the result of the sums of operators. For this purpose, we must verify the assumptions of Theorem 3.2.1. The spectral properties of A and B are as follows.

Proposition 3.4.1. *A and B are linear closed operators and their domains are dense in E. Moreover, they satisfy assumptions (3.3), (3.4) and (3.5).*

Proof. 1. Let us study the spectral equation related to the operator B

$$Bw + zw = k.$$

Fix some positive μ_0 and let z such that $Re\ z \geq \mu_0$. Then the general solution of the problem

$$\begin{cases} t^{2m\alpha} w'(t) + zw(t) &= k(t),\ t \in [0,1], \\ w(0) = 0, \end{cases}$$

is given by

$$w(t) = d \exp\left(z \int_t^1 \frac{ds}{s^{2m\alpha}}\right) + \int_0^t \left(\frac{k(\sigma)}{\sigma^{2m\alpha}} \exp\left(-z \int_\sigma^t \frac{ds}{s^{2m\alpha}}\right)\right) d\sigma,$$

where d is an arbitrary constant. The hypothesis $p > 1 + N\alpha$ implies that the function

$$t \mapsto t^{-2m\alpha + (N\alpha/p)} \exp\left(z \int_t^1 \frac{ds}{s^{2m\alpha}}\right)$$

does not belong to $L^p(B(0,1))$. So, we will take $d = 0$ to obtain $w \in E$. Consequently

$$\begin{aligned} w(t) &= \left((B + zI)^{-1} k\right)(t) \\ &= \int_0^t \left(\frac{k(\sigma)}{\sigma^{2m\alpha}} \exp\left(-z \int_\sigma^t \frac{ds}{s^{2m\alpha}}\right)\right) d\sigma \\ &= \exp\left(\tfrac{z}{(2m\alpha-1)t^{2m\alpha-1}}\right) \int_0^t \frac{k(\sigma)}{\sigma^{2m\alpha}} \exp\left(\tfrac{-z}{(2m\alpha-1)\sigma^{2m\alpha-1}}\right) d\sigma. \end{aligned}$$

Let us check that this formula is well defined on $[0,1]$ and gives $w(0) = 0$. Set $\mu = \frac{z}{(2m\alpha-1)}$, then

$$\begin{aligned}\|w(t)\| &\leq \exp\left(\tfrac{Re\,\mu}{t^{2m\alpha-1}}\right)\textstyle\int_0^t \left\|\sigma^{-2m\alpha+(N\alpha/p)}k(\sigma)\right\|\sigma^{-N\alpha/p}\exp\left(\tfrac{-Re\,\mu}{\sigma^{2m\alpha-1}}\right)d\sigma \\ &\leq \left(\textstyle\int_0^t \left\|\sigma^{-2m\alpha+(N\alpha/p)}k(\sigma)\right\|^p d\sigma\right)^{1/p}\left(\textstyle\int_0^t \sigma^{-qN\alpha/p}d\sigma\right)^{1/q} \\ &\leq \left(\tfrac{1}{1-(qN\alpha)/p}\right)^{\frac{1}{q}} t^{(1/q)-(N\alpha/p)}\|k\|_E,\end{aligned}$$

where $(1/p)+(1/q)=1$. Hence $w(t)$ is defined and $w(0)=0$ since

$$\frac{1}{q}-\frac{N\alpha}{p}=1-\frac{1}{p}-\frac{N\alpha}{p}$$

means $p>1+N\alpha$. On the other hand we can write

$$\begin{aligned}&t^{-2m\alpha+(N\alpha/p)}w(t)\\ &= \textstyle\int_0^t\left(\dfrac{k(\sigma)}{t^{2m\alpha-(N\alpha/p)}\sigma^{2m\alpha}}\exp\mu\left(t^{-2m\alpha+1}-\sigma^{-2m\alpha+1}\right)\right)d\sigma\\ &= \textstyle\int_0^t\left(\dfrac{k(\sigma)}{\sigma^{2m\alpha-(N\alpha/p)}}\left(\dfrac{1}{t^{2m\alpha-(N\alpha/p)}\sigma^{N\alpha/p}}\exp\mu\left(t^{-2m\alpha+1}-\sigma^{-2m\alpha+1}\right)\right)\right)d\sigma.\end{aligned}$$

Putting

$$K_\mu(t,\sigma)=\begin{cases}\dfrac{1}{t^{2m\alpha-(N\alpha/p)}\sigma^{N\alpha/p}}\exp\mu\left(t^{-2m\alpha+1}-\sigma^{-2m\alpha+1}\right) & \text{if } t>\sigma,\\ 0 & \text{if } t<\sigma,\end{cases}$$

we deduce that

$$t^{-2m\alpha+(N\alpha/p)}w(t)=\int_0^1\frac{k(\sigma)}{\sigma^{2m\alpha-(N\alpha/p)}}K_\mu(t,\sigma)\,d\sigma.$$

We need the following classical interpolation result, the so-called Schur's Lemma.

Lemma 3.4.1. *If there exists a constant C such that*

a) $\int_0^1|K_\mu(t,\sigma)|d\sigma\leq C$ *for every* $t\in\,]0,1[$,

b) $\int_0^1|K_\mu(t,\sigma)|dt\leq C$ *for every* $\sigma\in\,]0,1[$,

then

$$\left\|t^{-2m\alpha+(N\alpha/p)}w\right\|_{L^p(0,1;X)}\leq C\left\|t^{-2m\alpha+(N\alpha/p)}k\right\|_{L^p(0,1;X)}.$$

Now, we have to check that the conditions a) and b) are satisfied.

Condition a)

We have

$$\begin{aligned}
\int_0^1 |K_\mu(t,\sigma)|\, d\sigma &= \tfrac{1}{t^{2m\alpha-(N\alpha/p)}} \exp\left(t^{-2m\alpha+1}.Re\ \mu\right) \int_0^t \tfrac{\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)}{\sigma^{N\alpha/p}} d\sigma \\
&\leq \tfrac{1}{2m\alpha-1} \exp\left(t^{-2m\alpha+1}.Re\ \mu\right) \int_{t^{-2m\alpha+1}}^{+\infty} \exp\left(-s.Re\ \mu\right) ds \\
&\leq \tfrac{1}{Re\ z}.
\end{aligned}$$

Consequently

$$\max_{t\in[0,1]} \int_0^1 |K_\mu(t,\sigma)|\, d\sigma \leq \frac{1}{Re\ z}. \tag{3.11}$$

This shows that the condition a) of Lemma 3.4.1 holds true.

Condition b)

We have

$$\begin{aligned}
\int_0^1 |K_\mu(t,\sigma)|\, dt &= \tfrac{1}{\sigma^{\frac{N\alpha}{p}}} \exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right) \int_\sigma^1 \tfrac{\exp\left(t^{-2m\alpha+1}.Re\ \mu\right)}{t^{2m\alpha-(N\alpha/p)}} dt \\
&= \tfrac{1}{2m\alpha-1} \tfrac{\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)}{\sigma^{\frac{N\alpha}{p}}} \int_1^{\sigma^{-2m\alpha+1}} \tfrac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp\left(s.Re\ \mu\right) ds,
\end{aligned}$$

and

$$\begin{aligned}
\int_1^{\sigma^{-2m\alpha+1}} \tfrac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp\left(s.Re\ \mu\right) ds &= \int_1^{\frac{1+\sigma^{-2m\alpha+1}}{2}} \tfrac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp\left(s.Re\ \mu\right) ds \\
&\quad + \int_{\frac{1+\sigma^{-2m\alpha+1}}{2}}^{\sigma^{-2m\alpha+1}} \tfrac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp\left(s.Re\ \mu\right) ds \\
&\leq \int_1^{\frac{1+\sigma^{-2m\alpha+1}}{2}} \exp\left(s.Re\ \mu\right) ds \\
&\quad + \tfrac{1}{\left(\frac{1+\sigma^{-2m\alpha+1}}{2}\right)^{\frac{N\alpha}{p(2m\alpha-1)}}} \int_{\frac{1+\sigma^{-2m\alpha+1}}{2}}^{\sigma^{-2m\alpha+1}} \exp\left(s.Re\ \mu\right) ds, \\
&= I_1 + I_2.
\end{aligned}$$

Then

$$\begin{aligned}
I_1 &\leq \tfrac{1}{Re\ \mu}\left[\exp\left(Re\ \mu \tfrac{(1+\sigma^{-2m\alpha+1})}{2}\right) - \exp(Re\ \mu)\right] \\
&\leq \tfrac{1}{Re\ \mu} \exp\left(Re\ \mu \tfrac{(1+\sigma^{-2m\alpha+1})}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2m\alpha-1}\frac{\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)}{\sigma^{\frac{N\alpha}{p}}}I_1 &\leq \frac{1}{Re\ z}\frac{\exp(\frac{-(\sigma^{-2m\alpha+1}-1)}{2}.Re\ \mu)}{\sigma^{\frac{N\alpha}{p}}} \\
&\leq \frac{1}{Re\ z}\frac{\exp(\frac{-(\sigma^{-2m\alpha+1}-1)}{2}.\mu_0)}{\sigma^{\frac{N\alpha}{p}}} \\
&\leq \frac{C_1(\alpha,p)}{Re\ z},
\end{aligned}$$

since the function

$$\sigma \mapsto \frac{\exp(\frac{-(\sigma^{-2m\alpha+1}-1)}{2}.\mu_0)}{\sigma^{\frac{N\alpha}{p}}}$$

is continuous on $[0,1]$. Moreover

$$\begin{aligned}
\frac{1}{2m\alpha-1}\frac{\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)}{\sigma^{\frac{N\alpha}{p}}}I_2 &\leq \frac{1}{2m\alpha-1}\frac{\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)}{\sigma^{\frac{N\alpha}{p}}(\frac{1+\sigma^{-2m\alpha+1}}{2})^{\frac{N\alpha}{p(2m\alpha-1)}}}\int_{\frac{1+\sigma^{-2m\alpha+1}}{2}}^{\sigma^{-2m\alpha+1}}\exp\left(s.Re\ \mu\right)ds \\
&\leq \frac{1}{Re\ z}\frac{C_2(\alpha,p)}{\sigma^{\frac{N\alpha}{p}}(\frac{1+\sigma^{-2m\alpha+1}}{2})^{\frac{N\alpha}{p(2m\alpha-1)}}} \\
&\leq \frac{C_3(\alpha,p)}{Re\ z},
\end{aligned}$$

in virtue of the fact that

$$\lim_{\sigma\to 0}\frac{1}{\sigma^{\frac{N\alpha}{p}}\left(1+\sigma^{-2m\alpha+1}\right)^{\frac{N\alpha}{p(2m\alpha-1)}}} = 1.$$

Consequently, there exists some constant $C(\alpha,p)>0$ such that

$$\max_{\sigma\in[0,1]}\int_0^1\left|K_z\left(t,\sigma\right)\right|dt \leq \frac{C(\alpha,p)}{Re\ z}. \tag{3.12}$$

This shows that the condition b) of Lemma 3.4.1 holds also true. Now, using Lemma 3.4.1 together with (3.11) and (3.12), we obtain

$$\left\|t^{-2m\alpha+(N\alpha/p)}w\right\|_{L^p(0,1;X)} \leq \frac{C(\alpha,p)}{Re\ z}\left\|t^{-2m\alpha+(N\alpha/p)}k\right\|_{L^p(0,1;X)},$$

from which it follows

$$\left\|(B+zI)^{-1}\right\|_{L(E)} \leq \frac{C(\alpha,p)}{Re\ z}.$$

Thus, we can take $\theta_B=\frac{\pi}{2}-\theta_0$, (for each $\theta_0\in\left]0,\frac{\pi}{2}\right[$).

2. Now, we are concerned with the operator A which has the same properties as its realization $L\left(t\right)$. The study uses the following perturbation result due to Lunardi ([24, Proposition 2.4.3, p. 65]).

Proposition 3.4.2. *Let L_0 be a linear operator of domain $D(L_0)$ dense in E. Assume that L_0 is sectorial and P a linear continuous operator on $D(L_0)$ which is compact. Then operator $L_0 + P : D(L_0) \to X$ is sectorial.*

For each $t \in [0,1]$, we write

$$L(t)\psi \;=\; L_0\psi + P(t)\psi$$

with

$$\begin{cases} D(L_0) &= \left\{ \psi \in W^{2m,p}(B(0,1)) : \partial_\nu^l \psi\big|_{\partial B(0,1)} = 0,\ l = 0,1,...,m-1 \right\}, \\ L_0\psi &= M\psi = \sum_{k=1}^N \partial_{y_k}^{2m}\psi, \end{cases}$$

and

$$\begin{cases} D(P(t)) &= W^{1,p}(B(0,1)), \\ P(t)\psi &= -\alpha t^{2m\alpha-1}\sum_{k=1}^N y_k \partial_{y_k}\psi. \end{cases}$$

It is well known that $\overline{D(L_0)} = L^p(B(0,1))$. The fact that L_0 is sectorial can be proved as in [8, Lemma 5.2 and Lemma 5.3 , pp. 18-19]. Observe that

$$\psi^{(l)}(y) = \int_0^y (-s\psi^{(l+1)}(s))ds + \int_y^1 (1-s)\psi^{(l+1)}(s)ds - \int_0^1 \psi^{(l)}(s)ds;\ l = 1,2,...,2m-1,$$

where $\psi^{(l)}$ denotes the derivative of order l of ψ. Thanks to Hölder inequality, for $\psi \in D(L_0) \subset D(P(t))$ and by using the previous equality we have

$$\begin{aligned}
&\|P(t)\psi\|_{L^p(B(0,1))} \\
&\quad= \left(\int_{B(0,1)} \left|-\alpha t^{2m\alpha-1}\sum_{k=1}^N y_k\partial_{y_k}\psi(y_1,y_2,...,y_N)\right|^p dy_1dy_2...dy_N\right)^{\frac{1}{p}} \\
&\quad\le \sum_{k=1}^N \left(\int_{B(0,1)} \left|-\alpha t^{2m\alpha-1} y_k\left(\int_0^{y_k} s_k\partial_{s_k}^2\psi ds_k + \int_{y_k}^1 (1-s_k)\partial_{s_k}^2\psi ds_k\right)\right|^p dy_1...dy_N\right)^{\frac{1}{p}} \\
&\quad\le \alpha t^{2m\alpha-1}\left[C_1(p)\|M\psi\|_{L^p(B(0,1))} + C_2(p)\|M\psi\|_{L^p(B(0,1))}\right] \\
&\quad\le C_3(\alpha,p)\|\psi\|_{D(L_0)}.
\end{aligned}$$

On the other hand, let us set

$$\begin{array}{llll} m_k(t): & L^p(B(0,1)) & \rightarrow & L^p(B(0,1)) \\ & \psi & \mapsto & (m_k(t)\psi) = -\alpha t^{2m\alpha-1} y_k \psi,\ k=1,...,N, \\ & & & \\ i: & W^{1,p}(B(0,1)) & \rightarrow & L^p(B(0,1)) \\ & \psi & \mapsto & \psi, \\ & & & \\ d_k: & W^{2m,p}(B(0,1)) & \rightarrow & W^{1,p}(B(0,1)) \\ & \psi & \mapsto & d_k(\psi) = \partial_{y_k}\psi,\ k=1,...,N. \end{array}$$

Then one has

$$P(t) = \sum_{k=1}^{N} P_k(t) = \sum_{k=1}^{N} m_k(t) \circ i \circ d_k.$$

Thus, $P(t)$ is compact from $D(L_0)$ into E since i is compact and d_k, $m_k(t), k = 1, ..., N$ are continuous. So for any $t \in [0,1]$, the operator $L(t)$ is sectorial and consequently there exist some $r_0 > 0$ and $\theta_1 \in \left]0, \frac{\pi}{2}\right[$ such that

$$\rho(-L(t)) \supset \Sigma_{\pi-\theta_1} = \{z : |z| \geq r_0,\ \ |\arg z| < \pi - \theta_1\}.$$

Now, for $k \in E$ and $z \in \Sigma_{\pi-\theta_1}$ the spectral equation

$$Aw + zw = k,$$

is equivalent to

$$L(t)\, w(t) + zw(t) = k(t),\ t \in [0,1],$$

which admits a unique solution

$$w(t) = (L(t) + zI)^{-1} k(t).$$

Hence

$$\|w(t)\|_{L^p(B(0,1))} \leq \frac{K}{|z|} \|k(t)\|_{L^p(B(0,1))},$$

which implies

$$\|w\|_E = \left(\int_0^1 \left\| t^{-2m\alpha+(N\alpha/p)} w(t) \right\|_X^p d\tau \right)^{1/p} \leq \frac{K}{|z|} \|k\|_E.$$

This ends the proof of Proposition 3.4.1. $\square$

Proposition 3.4.3. *Operators A and B satisfy the Labbas-Terreni condition (3.6).*

Proof. In our case, since the domains $D(L(t))$ are constant, the condition (3.6) holds whenever the following so-called estimate of Sobolevskii [32] is fulfilled: There exists $K > 0$ such that for all $t, \sigma \in [0,1]$,

$$\left\|(L(t)L(\sigma)^{-1} - I)\right\|_{L(X)} \leq K \left|t-\sigma\right|^{\rho}. \tag{3.13}$$

For $g \in X = L^p(B(0,1))$, the equation $\psi = L(\sigma)^{-1}g$ is equivalent to

$$\begin{cases} (L\,(t)\,\psi)\,(y) & = & M\psi - \alpha t^{2m\alpha-1}\sum_{k=1}^{N} y_k \partial_{y_k}\psi(y) = g(y), \\ \partial_\nu^l \psi\big|_{\partial B(0,1)} & = & 0,\ l = 0,1,...,m-1, \end{cases}$$

and

$$[(L(t)-L(\sigma))L(\sigma)^{-1}g](y) = \alpha(\sigma^{2m\alpha-1} - t^{2m\alpha-1})\sum_{k=1}^{N} y_k \partial_{y_k}\psi(y),$$

where $y = (y_1, y_2, ..., y_N)$. Then, we get

$$\begin{array}{rcl} \|[(L(t)-L(\sigma))L(\sigma)^{-1}g]\|_X & \leq & \alpha\left|t^{2m\alpha-1} - \sigma^{2m\alpha-1}\right| \left\|\sum_{k=1}^{N} y_k \partial_{y_k}\psi\right\|_X \\ & \leq & M_1 \left|t-\sigma\right|^{\min(1,2m\alpha-1)} \left\|M\psi\right\|_{L^p(B(0,1))} \\ & \leq & M_2 \left|t-\sigma\right|^{\min(1,2m\alpha-1)} \left\|\psi\right\|_{W^{2m,p}(B(0,1))} \\ & \leq & K \left|t-\sigma\right|^{\min(1,2m\alpha-1)} \left\|g\right\|_{L^p(B(0,1))}. \end{array}$$

So, the condition (3.13) is satisfied with $\rho = \min\left(1, 2m\alpha - 1\right)$. To prove (3.6), it is sufficient to estimate

$$\left\|A\left(A+\lambda\right)^{-1}\left[A^{-1};(B+z)^{-1}\right]\right\|_{L(E)},$$

where $\lambda \in \rho(-A)$ and $z \in \rho(-B)$. Let $k \in E$, then

$$\begin{aligned}
D &= \left(t^{-2m\alpha+(N\alpha/p)} A(A+\lambda)^{-1}\left[A^{-1};(B+z)^{-1}\right]k\right)(t) \\
&= t^{-2m\alpha+(N\alpha/p)}(A(A+\lambda)^{-1}\left(A^{-1}(B+z)^{-1}-(B+z)^{-1}A^{-1}\right)k)(t) \\
&= t^{-2m\alpha+(N\alpha/p)} L(t)(L(t)+\lambda)^{-1} \\
&\quad \times \left[L(t)^{-1}\left((B+z)^{-1}k\right)(t)-\left((B+z)^{-1}L(t)^{-1}k\right)(t)\right] \\
&= L(t)(L(t)+\lambda)^{-1}\textstyle\int_0^1 \sigma^{-2m\alpha+(N\alpha/p)} K_\mu(t,\sigma)\left(L(t)^{-1}-L(\sigma)^{-1}\right)k(\sigma)\,d\sigma \\
&= \textstyle\int_0^1 \sigma^{-2m\alpha+(N\alpha/p)} K_\mu(t,\sigma)L(t)(L(t)+\lambda)^{-1}\left(L(t)^{-1}-L(\sigma)^{-1}\right)k(\sigma)\,d\sigma \\
&= \textstyle\int_0^1 \sigma^{-2m\alpha+(N\alpha/p)} K_\mu(t,\sigma)(L(t)+\lambda)^{-1}\left(I-L(t)L(\sigma)^{-1}\right)k(\sigma)\,d\sigma,
\end{aligned}$$

since the domains $D(L(t))$ are constant, where (we recall)

$$K_\mu(t,\sigma)=\begin{cases}\dfrac{1}{t^{2m\alpha-(N\alpha/p)}\sigma^{N\alpha/p}}\exp\mu\left(t^{-2m\alpha+1}-\sigma^{-2m\alpha+1}\right) & \text{if } t>\sigma, \\ 0 & \text{if } t<\sigma,\end{cases}$$

with $\mu=\frac{z}{(2m\alpha-1)}$. Then

$$\|D\|_X \leq \frac{K}{|\lambda|}\textstyle\int_0^1 |K_\mu(t,\sigma)|\,|t-\sigma|^\rho\,\sigma^{-2m\alpha+(N\alpha/p)}\,\|k(\sigma)\|_X\,d\sigma,$$

with $\rho=\min(1,2m\alpha-1)$. We have

$$\begin{aligned}
\textstyle\int_0^1 |K_\mu(t,\sigma)|\,|t-\sigma|^\rho\,d\sigma &= \frac{1}{t^{2m\alpha-(N\alpha/p)}}\exp\left(t^{-2m\alpha+1}.Re\ \mu\right) \\
&\quad \times \textstyle\int_0^t \sigma^{-N\alpha/p}(t-\sigma)^\rho\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)d\sigma.
\end{aligned}$$

Then by Hölder inequality, one has

$$\begin{aligned}
&\textstyle\int_0^t \sigma^{-N\alpha/p}(t-\sigma)^\rho\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)d\sigma \\
&\qquad\leq \left(\textstyle\int_0^t \sigma^{-N\alpha/p}\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)d\sigma\right)^{1-\rho} \\
&\qquad\quad \times\left(\textstyle\int_0^t \sigma^{-N\alpha/p}(t-\sigma)\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)d\sigma\right)^{\rho}
\end{aligned}$$

and

$$\begin{aligned} J_1 &= \left(\int_0^t \sigma^{2m\alpha-(N\alpha/p)}\sigma^{-2m\alpha}\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)d\sigma\right)^{1-\rho} \\ &\leq \frac{\left(t^{2m\alpha-(N\alpha/p)}\right)^{1-\rho}}{(2m\alpha-1)^{1-\rho}}\frac{1}{(Re\ \mu)^{1-\rho}}\left(\exp\left(-t^{-2m\alpha+1}.Re\ \mu\right)\right)^{1-\rho}, \\ J_2 &= \left(\int_0^t \sigma^{2m\alpha-(N\alpha/p)}\sigma^{-2m\alpha}(t-\sigma)\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)d\sigma\right)^{\rho} \\ &\leq \frac{\left(t^{2m\alpha-(N\alpha/p)}\right)^{\rho}}{(2m\alpha-1)^{\rho}}\frac{1}{(Re\ \mu)^{\rho}}\left(\int_0^t (t-\sigma)\chi'(\sigma)\,d\sigma\right)^{\rho}, \end{aligned}$$

where $\chi(\sigma) = \exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)$. Using an integration by parts, we obtain

$$\begin{aligned} \int_0^t (t-\sigma)\chi'(\sigma)\,d\sigma &= \int_0^t \exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)d\sigma \\ &= \int_0^t \sigma^{2m\alpha}\sigma^{-2m\alpha}\exp\left(-\sigma^{-2m\alpha+1}.Re\ \mu\right)d\sigma \\ &\leq \frac{t^{2m\alpha}}{2m\alpha-1}\frac{1}{Re\ \mu}\exp\left(-t^{-2m\alpha+1}.Re\ \mu\right), \end{aligned}$$

from which we deduce that

$$J_2 \leq \frac{(t^{2m\alpha-(N\alpha/p)})^{\rho}}{(2m\alpha-1)^{\rho}}\frac{1}{(Re\ \mu)^{\rho}}\frac{(t^{2m\alpha})^{\rho}}{(2m\alpha-1)^{\rho}}\frac{1}{(Re\ \mu)^{\rho}}\left(\exp\left(-t^{-2m\alpha+1}.Re\ \mu\right)\right)^{\rho}.$$

Finally we have

$$\begin{aligned} &\int_0^1 |K_\mu(t,\sigma)|\,|t-\sigma|^{\rho}\,d\sigma \\ &\qquad \leq \frac{\exp\left(t^{-2m\alpha+1}.Re\ \mu\right)}{t^{2m\alpha-(N\alpha/p)}}\frac{\left(t^{2m\alpha-(N\alpha/p)}\right)^{1-\rho}}{(2m\alpha-1)^{1-\rho}}\frac{\left(\exp\left(-t^{-2m\alpha+1}.Re\ \mu\right)\right)^{1-\rho}}{(Re\mu)^{1-\rho}} \\ &\qquad \times\frac{\left(t^{2m\alpha-(N\alpha/p)}\right)^{\rho}}{(2m\alpha-1)^{\rho}}\frac{1}{(Re\ \mu)^{\rho}}\frac{\rho}{(2m\alpha-1)^{\rho}}\frac{1}{(Re\ \mu)^{\rho}}\left[\exp\left(-t^{-2m\alpha+1}Re\ .\mu\right)\right]^{\rho} \\ &\qquad \leq \frac{(t^{2m\alpha})^{\rho}}{(2m\alpha-1)^{1+\rho}}\frac{1}{(Re\ \mu)^{1+\rho}}, \end{aligned}$$

and

$$\max_{t\in[0,1]}\int_0^1 |K_\mu(t,\sigma)|\,|t-\sigma|^{\rho}\,d\sigma \leq \frac{C}{(Re\ \mu)^{1+\rho}}. \tag{3.14}$$

In a similar manner we obtain

$$\max_{\sigma\in[0,1]}\int_0^1 |K_\mu(t,\sigma)|\,|t-\sigma|^{\rho}\,dt \leq \frac{C}{(Re\ \mu)^{1+\rho}}. \tag{3.15}$$

Now, using Schur interpolation Lemma together with (3.14) and (3.15), we obtain

$$\left\| A\left(A+\lambda\right)^{-1}\left[A^{-1};\left(B+z\right)^{-1}\right]\right\|_{L(E)} \leq \frac{C}{\left|\lambda\right|\left(Re\mu\right)^{1+\rho}} = \frac{C}{\left|\lambda\right|\left(Re\ z\right)^{1+\rho}},$$

which implies

$$\left\| A\left(A+\lambda\right)^{-1}\left[A^{-1};\left(B+z\right)^{-1}\right]\right\|_{L(E)} \leq \frac{C}{\left|\lambda\right|\left|z\right|^{1+\rho}},$$

for any $\lambda \in \rho\left(-A\right)$ and any z belonging to a simple path γ joining $\infty e^{-i\theta_2}$ to $\infty e^{i\theta_2}$ for some $\theta_2 \in]\pi-\theta_B,\theta_1[$, γ lies to $\Sigma_{\pi-\theta_1}\cap\Sigma_{\pi-\theta_B}$. Then (3.6) is verified with

$$\left(\tau,\rho\right) = \left(0,\min\left(1,2m\alpha-1\right)\right).$$

□

3.5 Regularity results for the original problem

3.5.1 Regularity results for the transformed problem (3.7)

Using Theorem 3.2.1, we deduce the following result

Proposition 3.5.1. *There exists λ^* such that for all $\lambda \geq \lambda^*$ and for all $k \in D_A\left(\sigma,p\right)$ (respectively, $k \in D_B\left(\sigma,p\right)$), Problem (3.10) admits a unique solution $w \in D\left(A\right)\cap D\left(B\right)$ such that for all $\theta \leq \min\left(\sigma,2m\alpha-1\right)$*

i) $L\left(.\right)w \in D_A\left(\theta,p\right)$,

ii) $t^\alpha w' \in D_A\left(\theta,p\right)$,

iii) $L\left(.\right)w \in D_B\left(\theta,p\right)$

(respectively,

i) $L\left(.\right)w \in D_B\left(\theta,p\right)$,

ii) $t^\alpha w' \in D_B\left(\theta,p\right)$,

iii) $L\left(.\right)w \in D_A\left(\theta,p\right)$*).*

Now, let us specify the space $D_A\left(\sigma,p\right)$. One has

$$D_A(\sigma,p)=$$
$$\begin{cases}\left\{w\in E: t^{-2m\alpha+(N\alpha/p)}w\in L^p\left(0,1;W^{2m\sigma,p}\left(B(0,1)\right)\right),\ \partial_\nu^l w\big|_{\partial B(0,1)}=0\right\}\\ \text{where } l=0,1,...,m-1;\ \text{if } 2m\sigma>1/p,\\ \\ \left\{w\in E: t^{-2m\alpha+(N\alpha/p)}w\in L^p\left(0,1;W^{2m\sigma,p}\left(B(0,1)\right)\right)\right\}\ \text{if } 2m\sigma<1/p.\end{cases}$$

Indeed, we know that

$$D_A(\sigma,p)=\left\{w\in E:\left\|\zeta^{1-\sigma}Ae^{-\zeta A}w\right\|_E\in L_*^p\right\},$$

because $(-A)$ is a generator of the analytic semigroup $\left\{e^{-\zeta A}\right\}_{\zeta\geq 0}$. Now, $w\in D_A(\sigma,p)$ implies

$$\left\|\zeta^{1-\sigma}Ae^{-\zeta A}w\right\|_E\in L_*^p.$$

Or $\left\|\zeta^{1-\sigma}Ae^{-\zeta A}w\right\|_E\in L_*^p$ is equivalent to

$$\begin{aligned}&\textstyle\int_0^\infty\left\|\zeta^{1-\sigma}Ae^{-\zeta A}w\right\|_E^p\frac{d\zeta}{\zeta}\\ &\quad=\textstyle\int_0^\infty\left\|t^{-2m\alpha+(N\alpha/p)}\zeta^{1-\sigma}Ae^{-\zeta A}w\right\|^p_{L^p(0,1;L^p(B(0,1)))}\dfrac{d\zeta}{\zeta}\\ &\quad=\textstyle\int_0^\infty\left(\int_0^1\left\|t^{-2m\alpha+(N\alpha/p)}\zeta^{1-\sigma}\left(Ae^{-\zeta A}w\right)(t)\right\|^p_{L^p(B(0,1))}dt\right)\dfrac{d\zeta}{\zeta}<+\infty.\end{aligned}$$

On the other hand, thanks to the Dunford representation of the semigroup $\left\{e^{-\zeta A}\right\}_{\zeta\geq 0}$, we have

$$e^{-\zeta A}=\frac{1}{2i\pi}\int_\gamma e^{\zeta\lambda}\left(A+\lambda I\right)^{-1}d\lambda,$$

where γ is a sectorial curve lying in $\rho(-A)$ such that $Re(-\lambda)<0$ for a larger $\lambda\in\gamma$. Moreover

$$\left(Ae^{-\zeta A}w\right)(t)=L(t)\,e^{\zeta L(t)}\left(w(t)\right).$$

Then, by Fubini's Theorem, we obtain

$$\begin{aligned}&\textstyle\int_0^\infty\left\|\zeta^{1-\sigma}Ae^{-\zeta A}w\right\|_E^p\frac{d\zeta}{\zeta}\\ &\quad=\textstyle\int_0^\infty\left[\int_0^1\left\|t^{-2m\alpha+(N\alpha/p)}\zeta^{1-\sigma}L(t)\,e^{\zeta L(t)}\left(w(t)\right)\right\|^p_{L^p(B(0,1))}dt\right]\dfrac{d\zeta}{\zeta}\\ &\quad=\textstyle\int_0^1\left\|t^{-2m\alpha+(N\alpha/p)}\right\|^p\left[\int_0^\infty\left\|\zeta^{1-\sigma}L(t)\,e^{\zeta L(t)}w(t)\right\|^p_{L^p(B(0,1))}\dfrac{d\zeta}{\zeta}\right]dt<+\infty,\end{aligned}$$

which means that, for almost every t, the function

$$(y_1, y_2, ..., y_N) \quad \mapsto \quad t^{-2m\alpha+(N\alpha/p)}(t) w(t)(y_1, y_2, ..., y_N)$$

is in $D_{L(t)}(\sigma, p)$. It is well known that this last space is the following:

$$D_{L(t)}(\sigma, p) = \left(W^{2m,p}(B(0,1)) \cap W_0^{m,p}(B(0,1)); L^p(B(0,1))\right)_{1-\sigma,p},$$

and

$$\begin{aligned} &\left(W^{2m,p}(B(0,1)) \cap W_0^{2,p}(B(0,1)); L^p(B(0,1))\right)_{1-\sigma,p} \\ &= \begin{cases} \left\{w \in W^{2m\sigma,p}(B(0,1)) : \partial_\nu^l w\big|_{\partial B(0,1)} = 0,\ l = 0, 1, ..., m-1\right\} \\ \text{if } 2m\sigma > 1/p, \\ \\ W^{2m\sigma,p}(B(0,1)) \ \text{ if } 2m\sigma < 1/p. \end{cases} \end{aligned}$$

Let σ be a fixed positive number satisfying $\sigma < 1/2mp$ and $\sigma \leq 2m\alpha - 1$. From the above proposition, we deduce the following result.

Proposition 3.5.2. *For all h such that $t^{-2m\alpha+(N\alpha/p)} h \in L^p(0, 1; W^{2m\sigma,p}(B(0,1)))$, Problem (3.9) admits a unique solution fulfilling the following regularity properties:*

(i) $w \in L^p(]0,1[\times B(0,1))$, $t^{-2m\alpha+(N\alpha/p)} w \in L^p(]0,1[\times B(0,1))$, $w(0) = 0$,

(ii) $t^{-2m\alpha+(N\alpha/p)} Mw \in L^p(]0,1[\times B(0,1))$,

(iii) $t^{N\alpha/p} \partial_t w \in L^p(]0,1[\times B(0,1))$,

(iv) $t^{-2m\alpha+(N\alpha/p)} Mw \in L^p(0, 1; W^{2m\sigma,p}(B(0,1)))$,

(v) $t^{N\alpha/p} \partial_t w \in L^p(0, 1; W^{2m\sigma,p}(B(0,1)))$.

3.5.2 Going back to the original problem (3.1)

We go back to our original domain Q by using the inverse change of variables

$$\begin{aligned} \Pi^{-1} \quad &: G =]0,1[\times B(0,1) \quad \longrightarrow \quad Q \\ (t, y_1, y_2, ..., y_N) \quad &\longmapsto \quad (t, x_1, x_2, ..., x_N) = (t, t^\alpha y_1, t^\alpha y_2, ..., t^\alpha y_N). \end{aligned}$$

Let us recall that

$$\begin{cases} h(t, y_1, y_2, ..., y_N) = t^{2m\alpha} g(t, y_1, y_2, ..., y_N), \\ \\ g(t, y_1, y_2, ..., y_N) = f(t, x_1, x_2, ..., x_N), \\ \\ w(t, y_1, y_2, ..., y_N) = u(t, x_1, x_2, ..., x_N). \end{cases}$$

First, we see that

$$\begin{cases} \partial_{y_k} w = t^{\alpha} \partial_{x_k} u, \ k = 1, 2, ..., N, \\ \\ \partial_{y_k}^{2m} w = t^{2m\alpha} \partial_{x_k}^{2m} u, \ k = 1, 2, ..., N, \\ \\ \partial_t w = \partial_t u + (\alpha/t) \sum_{k=1}^{N} x_k \partial_{x_k} u. \end{cases}$$

The assumption $t^{-2m\alpha+(N\alpha/p)} h \in L^p(0,1; W^{2m\sigma,p}(B(0,1)))$ means that

$$\int_0^1 \left\| t^{-2m\alpha+(N\alpha/p)}(t) h(t,.) \right\|_{W^{2m\sigma}(B(0,1))}^p dt < \infty.$$

So, by setting

$$y = (y_1, y_2, ..., y_N), \ y' = (y_1', y_2', ..., y_N'), \ dy = dy_1 ... dy_N, \ dy' = dy_1' ... dy_N',$$

we have

$$\begin{aligned} &\textstyle\int_0^1 \left\| t^{-2m\alpha+(N\alpha/p)}(t) h(t,.) \right\|_{W^{2m\sigma}(B(0,1))}^p dt \\ &\quad = \int_0^1 t^{N\alpha-2m\alpha p} \int_{B(0,1)} \int_{B(0,1)} \frac{|h(t,y) - h(t,y')|^p}{\|y-y'\|^{2m\sigma p+N}} dy dy' dt \\ &\quad = \int_0^1 t^{2m\sigma\alpha p} \int_{\Omega_t} \int_{\Omega_t} \frac{|f(t,x) - f(t,x')|^p}{\|x-x'\|^{2m\sigma p+N}} dx dx' dt, \end{aligned}$$

where

$$x = (t^{\alpha} y_1, t^{\alpha} y_2, ..., t^{\alpha} y_N), \ x' = (t^{\alpha} y_1', t^{\alpha} y_2', ..., t^{\alpha} y_N'), \ dx = dx_1 ... dx_N, \ dx' = dx_1' ... dx_N',$$

and

$$\Omega_t = \left\{ (x_1, x_2, ..., x_N) \in \mathbb{R}^N : 0 \leq \sqrt{x_1^2 + x_2^2 + ... + x_N^2} < t^{\alpha} \right\}.$$

Let us introduce the following subspace of $L^p(Q)$:

$$L^p_{t^{2m\sigma\alpha}}\left(0,1; W^{2m\sigma,p}_{t^{\alpha}}\right) = \left\{ f \in L^p(Q) : \int_0^1 t^{2m\alpha\sigma p} \int_{\Omega_t} \int_{\Omega_t} \frac{|f(t,x) - f(t,x')|^p}{\|x-x'\|^{2m\sigma p+N}} dx dx' dt < \infty \right\}.$$

Then, we are in position to prove the main result of this work.

Theorem 3.5.1. *For given $\sigma \in]0,1[$ such that $0 < \sigma < \dfrac{1}{2mp}$ (such that p verifies (3.2)), and for any $f \in L^p_{t^{2m\alpha\sigma}}\left(0,1; W^{2m\sigma,p}_{t^{\alpha}}\right)$, Problem (3.1) has a unique solution $u \in H_p^{1,2m}(Q)$ with the regularities: $u,\ \partial_t u,\ \partial_{x_k} u,\ k = 1, ..., N$ and Mu belong to $L^p_{t^{2m\alpha\sigma}}\left(0,1; W^{2m\sigma,p}_{t^{\alpha}}\right)$.*

The proof of Theorem 3.5.1 can be easily deduced from the following equivalences.

Proposition 3.5.3. *(i)* $t^{-2m\alpha+(N\alpha/p)}h \in L^p\left(0,1;W^{2m\sigma,p}\left(B(0,1)\right)\right)$ *if and only if* $f \in L^p_{t^{2m\alpha\sigma}}\left(0,1;W^{2m\sigma,p}_{t^\alpha}\right)$,

(ii) $t^{-2m\alpha+(N\alpha/p)}w \in L^p\left(0,1;L^p\left(B(0,1)\right)\right)$ *if and only if* $u \in L^p\left(Q\right)$,

(iii) $t^{-2m\alpha+(N\alpha/p)}Mw \in L^p\left(0,1;W^{2m\sigma,p}\left(B(0,1)\right)\right)$ *if and only if* $Mu \in L^p_{t^{2m\alpha\sigma}}\left(0,1;W^{2m\sigma,p}_{t^\alpha}\right)$,

(iv) $t^{N\alpha/p}\partial_t w \in L^p\left(0,1;W^{2m\sigma,p}\left(B(0,1)\right)\right)$ *if and only if* $\partial_t u \in L^p_{t^{2m\alpha\sigma}}\left(0,1;W^{2\sigma,p}_{t^\alpha}\right)$.

Conclusion and prospects

In this work, we have studied a linear 2m-th order parabolic equation, on a time-varying domain of $\mathbb{R}^{N+1}$, subject to Dirichlet type condition on the lateral boundary, where the right-hand side of the equation is taken in the Lebesgue space L^p.

The approach is based on the use of the operators' sum method in Banach spaces; we have used the results of operators' sum theory in the non-commutative case.

We were particularly interested in the question of which sufficient conditions, as weak as possible, the dimension N, the exponent p and the type of the domain must be verified in order that our problem has a solution with optimal regularity.

This work may be extended at least in the following directions:

1. The high order operator M may be replaced by the following constant coefficient operator:

$$L = \sum_{|\delta|=|\beta|=m} (-1)^m a_{\delta\beta} \partial^\delta \partial^\beta$$

with $a_{\delta\beta} = a_{\beta\delta}$ and there exists a constant $C > 0$ such that

$$a_{\delta\beta}\xi^\delta\xi^\beta > C|\xi|^{2m},\ \xi \in \mathbf{R}^N.$$

2. The function f on the right-hand side of the equation of Problem (3.1), may be taken in Hölder or little Hölder spaces.

Bibliography

[1] E. A. Baderko, *The solvability of boundary value problems for higher order parabolic equations in domains with curvilinear lateral boundaries,* Differ. Uravn., **vol. 10**, no. 12, (1976), 1781-1792.

[2] R.M. Brown and W. Hu, *Boundary value problems for higher order parabolic equations,* Transactions of the american mathematical society, **vol. 353**, no. 2, (2000), 809-838.

[3] M. F. Cherepova, *On the solvability of boundary value problems for a higher order parabolic equation with growing coefficients,* Dokl. Math., **vol. 74**, no. 3, (2006), 819-820.

[4] S. Cherfaoui, A. Kessab and A. Kheloufi, *Study of a 2m-th order parabolic equation in non-symmetric conical domains,* Hacet. J. Math. Stat., vol. Accepted, doi: doi.org/10.15672/hujms.546340.

[5] S. Cherfaoui, A. Kessab and A. Kheloufi, *On 2m-th order parabolic equations with mixed boundary conditions in non-rectangular domains,* Sib. Èlektron. Mat. Izv., **vol. 14**, (2017), 73-91.

[6] S. Cherfaoui, A. Kessab, and A. Kheloufi, *Well-posedness and regularity results for a 2m-th order parabolic equation in symmetric conical domains of* $\mathbf{R}^{N+1}$, Mathematical Methods in the Applied Sciences, **vol. 40**, (2017), 6035-6047.

[7] G. Da Prato and P. Grisvard, *Sommes d'opérateurs linéaires et equations différentielles opérationnelles,* J. Math. Pures Appl., vol. IX Ser., no. 54, (1975), 305-387.

[8] M. Di Cristo, D. Guidetti and A. Lorenzi, *Abstract parabolic equations with applications to problems in cylindrical space domains,* Advances in Differential Equations, **vol. 15**, no. 1-2, (2010), 1-42.

[9] G. Dore and A. Venni, *On the Closedness of the Sum of Two Closed Operators,* Math. Z. 196 (1987), 189-201.

[10] A. Favini and A. Yagi, *Degenerate Differential Equations in Banach Spaces,* M. Dekker, New York, 1999.

[11] S. Gala, Q. Liu, and M. A. Ragusa, *Logarithmically improved regularity criterion for the nematic liquid crystal flows in $B^{-1}_{\infty,\infty}$ space,* Computers and Mathematics with Applications, **vol. 65**, (2013), 1738-1745.

[12] S. Gala and M. A. Ragusa, *A new regularity criterion for the 3D incompressible MHD equations via partial derivatives,* Journal of Mathematical Analysis and Applications, vol. 481, no. 2, Article number 123497, 2020.

[13] V. A. Galaktionov, *On regularity of a boundary point for higher-order parabolic equations: towards Petrovskii-type criterion by blow-up approach,* Nonlinear Di er. Equ. Appl., **vol. 16**, no. 5, (2009), 597-655.

[14] A. Grimaldi Piro and F. Ragnedda, *Higher order parabolic operators in domains with a "nonsmooth" boundary,* Rend. Sem. Fac.Sci. Univ. Cagliari, **vol. 54**, (1984), 45-62.

[15] P. Grisvard, *Caracterisation de quelques espaces d'interpolation,* Arch. Rational Mech. Anal., **vol. 25**, no. 1, (1967), 40-63.

[16] A. Kheloufi, *Intermediate regularity results for the solution of a high order parabolic equation,* Miskolc Mathematical Notes, **vol. 20**, no. 2, (2019), 1013-1019.

[17] A. Kheloufi and B. K. Sadallah, *Resolution of a high-order parabolic equation in conical time-dependent domains of* $\mathbb{R}^3$*,* Arab Journal of Mathematical Sciences, **vol. 22**, (2016), 165-181.

[18] R. Labbas, A. Medeghri and B. K. Sadallah, *On a parabolic equation in a triangular domain,* Applied Mathematics and Computation, **vol. 130**, (2002), 511-523.

[19] R. Labbas and B. Terreni, *Sommes d'opérateurs linéaires de type parabolique. lère partie,* Boll. Un. Mat. Ital., **vol. 1-B**, no. 7, (1987), 141-162.

[20] R. Labbas and B. Terreni, *Sommes d'opérateurs linéaires de type parabolique. 2ème partie,* Boll. Un. Mat. Ital., **vol. 2-B**, no. 7, (1988), 545-569.

[21] R. Labbas, *Some results on the sum of linear operators with non dense domain,* Annali di Matematica Pura ed Applicata, (IV), 154 (1989), 91-97.

[22] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type,* Translated from the Russian by S. Smith. Translations of Mathematical Monographs: Vol. 23 American Mathematical Society, Providence, R.I., 1968.

[23] J. L. Lions and E. Magenes, *Problèmes aux Limites Non Homogènes et Applications 1 , 2,* Paris, Dunod, 1968.

[24] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems,* Boston, Birkhäuser, 1995.

[25] V. P. Mikhailov, *The Dirichlet problem for a parabolic equation, I,* Mat. Sb., **vol. 61**, no. 103, (1963), 40-64.

[26] V. P. Mikhailov, *The Dirichlet problem for a parabolic equation, II,* Mat. Sb., **vol. 62**, no. 104, (1963), 140-159.

[27] M. A. Ragusa, *Cauchy-Dirichlet problem associated to divergence form parabolic equations,* Communications in Contemporary Mathematics, **vol. 6**, no. 3, (2004), 377-393

[28] M. A. Ragusa and A. Scapellato, *Mixed Morrey spaces and their applications to partial differential equations,* Nonlinear Analysis-Theory Methods and Applications, **vol. 151**, (2017), 51-65.

[29] M. A. Ragusa and A. Tachikawa, *Regularity for minimizers for functionals of double phase with variable exponents,* Adv. Nonlinear Anal., no. 9, (2020), 710-728.

[30] N. Roidos, *Closedness and invertibility for the sum of two operators,* Adv. Oper. Theory, no. 3, (2018), 582-605.

[31] B. Sadallah, *Etude d'un problème* 2*m-parabolique dans des domaines plan non rectangulaires,* Bollettino della Unione Matematica Italiana, **vol. 2B**, no. 5, (1983), 51-112.

[32] P. E. Sobolevskii, *On equations of parabolic type in Banach space,* Amer. Math. Soc. Transl., (1965), 1-62.

[33] H. Triebel, *Interpolation Theory, Function Spaces,* Differential Operators, North Holland, 1978.

[34] P. Grisvard, *Commutativité de deux foncteurs d'interpolation et applications,* J. Math. Pures Appl., 45 (1966), 143-290.

Abstract

In this thesis, we have studied a linear parabolic equation of any even order, on a domain of $\mathbb{R}^{N+1}$, with Dirichlet type condition on the lateral boundary, where the right-hand side of the equation is taken in the Lebesgue space L^p.

The study is based on the use of the results of operators' sum theory, non-commutative case, in Banach spaces.

We were interested in determining the sufficient conditions, as weak as possible, on the dimension N, the exponent p and the type of the domain which must be verified so that our problem has a solution with optimal regularity.

Résumé

Dans cette thèse, nous avons étudié une équation parabolique linéaire d'ordre pair quelconque, sur un domaine de $\mathbb{R}^{N+1}$, sous une condition aux limites de type Dirichlet sur la frontière latérale, où le membre droit de l'équation est pris dans l'espace de Lebesgue L^p.

L'étude est basée sur l'utilisation des résultats de la théorie des sommes d'opérateurs, cas non commutatif, dans les espaces de Banach.

Nous étions intéressés à déterminer les conditions suffisantes, aussi faibles que possible, sur la dimension N, l'exposant p et le type du domaine qui doivent être vérifiées pour que notre problème ait une solution avec une régularité optimale.

ملخص

درسنَا في هذه الأَطروحة نوعَا من أَنوَاع المعَادلَات الخطية المكَافِئة ذَات درجة زوجية كيفية علَى ميدَان من $\mathbb{R}^{n+1}$ ، تحت الشرط الحدي لـ دريكلي علَى الحَافة الجَانبية لميدَان التعريف، حيث يتم أَخذ الطرف الأَيمن لِلمعَادلة من فضَاء لوبيغ L^p .

تعتمد الدرَاسة علَى استخدَام نتَائِج نظرية مجموع المؤُثرَات - الحَالة غير التبديلية - في فضَاءَات بنَاخ.

كنَا مهتمين خَاصة بتحديد الشروط الكَافية، الضعيفة قدر الإِمكَان، علَى البعد n وَالقوة p ونوع المجَال، التي يجب استيفَاؤُهَا حتَى يكون لهذه المسأَلة حل مع إِنتظَام مثَالي.